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Can splits create market liquidity? Theory and evidence☆

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Abstract

We present a market microstructure model of stock splits in the presence of minimum tick size rules. The key feature of the model is that discretionary trading is endogenously determined. There exists a tradeoff between adverse selection costs on the one hand and discreteness related costs and opportunity costs of monitoring the market on the other hand. Under certain parameter values, there exists an optimal price. We document an inverse relation between the coefficient of variation of intraday trading volume and the stock price level. This empirical evidence and other existing evidence are consistent with the model. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

U.S. firms split their stocks quite frequently. In spite of inflation, positive real interest rates, and significant risk premiums, the average nominal stock price in the U.S. during the past 50 years has been almost constant. Why would firms keep on splitting their stocks to maintain low prices? This behavior is puzzling since, by doing so, firms actively increase their effective tick size (i.e., tick size/price), potentially exposing their stockholders to larger transaction costs.

This paper presents a value maximizing market microstructure model of stock splits. Our model joins practitioners in predicting that firms split their stocks to move the stock price into an optimal trading range in order to improve liquidity.[1,2] The driving force of the model stems from the fact that prices on U.S. exchanges are restricted to multiples of 1/8th of a dollar.[3] This restriction on prices creates a wedge between the "true" equilibrium price and the observed price.[4] Thus a portion of the transaction costs incurred by traders is purely an artifact of discreteness.

Anshuman and Kalay (1998) show that discreteness related commissions depend on the location of the "true" equilibrium price on the real line. In other words, whether the discrete pricing restriction is binding or not depends on the location of the "true" equilibrium price relative to a legitimate price (tick) in a discrete price economy. It may so happen that the "true" equilibrium price (plus any transaction cost) is close to a tick. Discreteness related commissions would be low in such a period. As information arrives in the market, the location of the "true" equilibrium price changes, and discreteness related commissions would, therefore, vary over time. They could be as low as 0 or as high as the tick size.

Interestingly, liquidity traders can take advantage of the variation in discreteness related commissions by timing their trades. Of course, such

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[1] Academicians have mostly relied on signaling models to explain stock splits (Grinblatt et al., 1984). More recently, Muscarella and Vetsuypens (1996) provide evidence consistent with the liquidity motive of stock splits. Practitioners, however, have all along held the belief that stock splits help restore an optimal trading range that maximizes the liquidity of the stock (see Baker and Powell, 1992; Bacon and Shin, 1993).

[2] Independent of our work, Angel (1997) has also presented a model of optimal price level that explains stock splits. In his model, the optimal price provides a tradeoff between firm visibility and transaction costs. In contrast, our model examines the behavior of liquidity traders in the presence of discrete pricing restrictions.

[3] There are exceptions to this restriction and more recently the NYSE has initiated a move toward decimal trading.

[4] The "true" equilibrium price is the market value of the asset conditional on all publicly available information in an otherwise identical continuous-price economy without any frictions (transaction costs).

strategic behavior is not costless. It involves close monitoring of the market to take advantage of periods with low discreteness related commissions. In general, liquidity traders differ in terms of their opportunity costs of monitoring the market. Some liquidity traders may prefer not to time the market because the benefits from timing trades do not offset their opportunity costs of monitoring. In contrast, other liquidity traders who are endowed with low opportunity costs of monitoring may find it beneficial to time their trades. Such discretionary traders would trade together in a period of low discreteness related commissions. The presence of additional liquidity traders in this period (a period of concentrated trading) forces the competitive market maker to charge a lower adverse selection commission than otherwise. Thus, discretionary liquidity traders save on execution costs – adverse selection as well as discreteness related commissions.

Because the tick size is fixed in nominal terms (at 1/8th of a dollar), the economic significance of the savings in discreteness related commissions depends on the stock price level. At low stock price levels, the savings in execution costs due to timing of trades may be significant enough to offset the opportunity costs of monitoring of most liquidity traders. There would be highly concentrated trading at low price levels as most liquidity traders would exercise the flexibility of timing trades. Conversely, at high stock price levels, few liquidity traders would time trades because the potential savings in execution costs are economically insignificant.

The key implication of the model is that the stock price level affects the distribution of liquidity trades across time, and consequently, the transaction costs incurred by them. In particular, we show that there exists an optimal stock price level that induces an optimal amount of discretionary trading. This optimal price results in the lowest (total) expected transaction costs incurred by all liquidity traders.

Because investors desire liquidity (Amihud and Mendelson, 1986; Brennan and Subrahmanyam, 1995), a value-maximizing firm should choose a stock price level that maximizes liquidity (minimizes the total transaction costs incurred by all liquidity traders). By splitting (or reverse splitting) its stock, a firm can always reset its stock price to the optimal price level.

We present numerical solutions of the model to show that, under certain parameter values, an optimal price exists. The numerical solutions show that the optimal price is increasing in the volatility of the underlying asset and decreasing in the fraction of liquidity traders. We also show that the optimal price is (linearly) increasing in the tick size. Finally, using intraday transaction data, we document a cross-sectional inverse relation between the coefficient of variation of time-aggregated trading volume (a measure of the degree of concentrated trading in a stock) and the stock price level. This empirical evidence and other existing evidence are consistent with the model.

The paper is organized as follows. Section 2 discusses a numerical example that illustrates the key features of the model. The model is developed in Section 3. Section 4 presents numerical solutions of the model. Section 5 discusses empirical evidence relevant to the model, and Section 6 concludes the paper.

2. A numerical example

Consider the following example that illustrates the central theme of the model – endogenization of discretionary trading. We make the following simplifying assumptions in the numerical example. (i) There are two trading opportunities (Periods 1 and 2). (ii) Discreteness related commissions in each period are either \$0.02 or \$0.10 with equal probability.[5] (iii) Firms are restricted to choose between two base prices (\$50 or \$100) – the base price could be thought of as the offer price in an initial public offering. (iv) Liquidity traders are of two types: 80 liquidity traders face very low opportunity costs of monitoring (\$0.01 per dollar of trade) and 40 liquidity traders face extremely high opportunity costs of monitoring. (v) In each period, there are a fixed number of informed traders who speculate on information that is revealed at the end of the period.

Before the market opens, liquidity traders face a strategic choice. They know that monitoring the market can help them time their trades into the period with low discreteness related commissions (\$0.02). Not only would they be saving on discreteness related commissions but also on adverse selection commissions because of the concentration of liquidity trades in a single period.

However, monitoring the market is not costless. Among the liquidity traders, those with extremely high monitoring costs would not find timing trades worthwhile. Such liquidity traders (40) behave like nondiscretionary traders. Assuming that there are negligible waiting costs, these traders would be indifferent between trading in Period 1 or trading in Period 2. Let equal number of nondiscretionary traders ($40/2=20$) arrive in the market in each period.

The interesting question is with regard to the 80 liquidity traders with low monitoring costs. Should they incur monitoring costs and time their trades or join the bandwagon of nondiscretionary traders? If they choose not to monitor (and, therefore, act as nondiscretionary traders), then each trading period would consist of $(80+40)/2=60$ liquidity traders, assuming that the arrival rate of nondiscretionary traders is constant (equal) in both periods. On the other hand, if these liquidity traders choose to monitor, one of the trading

[5] This assumption is purely for illustration purposes. In reality, there exists a probability distribution of discreteness related commissions over the interval (0, tick size).

periods would have 100 (80 discretionary and 20 nondiscretionary) liquidity traders, and the other period would have only 20 nondiscretionary liquidity traders. Hence the distribution of liquidity traders across the two periods would be one of the following: (60, 60) if they choose not to monitor the market and either (20, 100) or (100, 20) if they monitor the market.

Liquidity traders with low monitoring costs would think as follows. Their choice to monitor or not depends on the total (per dollar) transaction costs they face under each scenario. Total transaction costs are composed of adverse selection commissions, discreteness related commissions, and monitoring costs. Table 1 presents these costs at the two base prices in this economy.

Consider Panel A of Table 1 for the case when the base price is \$50. Suppose liquidity traders with low monitoring costs choose to monitor the market. Then, in the period they trade, the adverse selection commissions would be low because of the presence of 100 liquidity traders. In contrast, when they choose not to monitor the market, the adverse selection commissions are going to be higher because there would be only 60 liquidity traders. Assume that the adverse selection commissions are \$0.046 when there are 100 liquidity traders and \$0.535 when there are 60 liquidity traders (in the model, we derive the adverse selection commissions endogenously). Monitoring the market and concentrating trades in a single period results in savings of (\$0.535− \$0.046) = \$0.489 in adverse selection commissions, or 0.978% of the base price of \$50.

Panel B of Table 1 shows the adverse selection commissions when the base price is \$100. These numbers are scaled up versions of the adverse selection commissions when the base price is \$50. However, as shown in the (%) adverse selection commission column, the adverse selection commissions (given a fixed number of liquidity trades) are identical at both base prices in percentage terms. Therefore, the benefit of concentrated trading (in terms of savings in adverse selection commissions) is 0.978%, which is invariant to the base price.

Now consider discreteness related commissions when the base price is \$50 (Panel A). If liquidity traders with low monitoring costs choose to monitor, they would incur lower discreteness related commissions because they can time their trades in the period with low discreteness related commissions (\$0.02). Note that they would incur expected discreteness related commissions of \$0.04 (this is higher than \$0.02 because it is always possible that both trading periods have a realized discreteness related commission of \$0.10).[6] In contrast, when such liquidity traders choose not to monitor, they incur a higher expected discreteness related commission of \$0.06 (an average of \$0.02 and \$0.10). These commissions (\$ values) stay the same at the higher base price of \$100 (Panel B).

[6] The probability of both trading periods having high discreteness related commissions (\$0.10) is $0.5 \times 0.5 = 0.25$. The probability of at least one period having low discreteness related commissions (\$0.02) is $1 - 0.25 = 0.75$. Therefore, the expected discreteness related commissions is $0.25 \times \$0.10 + 0.75 \times \$0.02 = \$0.04$.

Table 1
Numerical example

Liquidity traders are of two types – those who incur low monitoring costs (80) and those who incur high monitoring costs (40). This numerical example illustrates the decision-making of liquidity traders with low monitoring costs. If these liquidity traders choose to monitor the market, the number of liquidity traders across the two periods would either be (100, 20) or (20, 100). If they choose not to monitor, the number of liquidity traders in each period would be 60. At a base price of \$50, it is better to monitor because the total transaction costs are lower (Panel A). Conversely, at a base price of \$100, it is better not to monitor (Panel B). The total transaction costs incurred by all liquidity traders (nondiscretionary and discretionary) is shown in Panel C.

Panel A: Decision to monitor the market (base price is \$50)

		Adverse selection commissions		Discreteness related commissions		Monitoring costs (per dollar)	Total transaction costs (per dollar)
Monitor	Liquidity traders	(\$)	(%)	(\$)	(%)	(%)	(%)
Yes	100	0.046	0.092	0.04	0.080	1.000	1.172
No	60	0.535	1.070	0.06	0.120	0.000	1.190
Savings		0.489	0.978	0.02	0.040	−1.000	0.018

Panel B: Decision to monitor the market (base price is \$100)

		Adverse selection commissions		Discreteness related commissions		Monitoring costs	Total transaction costs
Monitor	Liquidity traders	(\$)	(%)	(\$)	(%)	(%)	(%)
Yes	100	0.092	0.092	0.04	0.040	1.000	1.132
No	60	1.07	1.070	0.06	0.060	0.000	1.130
Savings		0.978	0.978	0.02	0.020	−1.000	−0.002

Panel C: Total transaction costs incurred by ALL liquidity traders

Base price	Distribution of trades	Adverse selection costs	Discreteness related costs	Monitoring costs	Total transaction costs
\$50	(100, 20) or (20, 100)	0.322	0.112	0.800	1.234
\$100	(60, 60)	1.284	0.072	0.000	1.356

Because discreteness causes fixed costs, the benefit of timing trades (due to savings in discreteness related commissions) is fixed at \$0.06−\$0.04 = \$0.02 independent of the base price. However, on a per dollar basis, the savings from timing trades are 0.04% at the lower base price of \$50, but only 0.02% at the higher base price of \$100.

Besides adverse selection commissions and discreteness related commissions, liquidity traders also incur monitoring costs (1%) if they choose to monitor. When the base price is \$50 (Panel A), the sum of adverse selection commissions, discreteness related commissions and monitoring costs is 1.172% upon monitoring and 1.19% without monitoring. When the base price is \$100 (Panel B), the total transaction costs are 1.132% upon monitoring and 1.130% without monitoring.

The decision to monitor or not depends on the total savings in transaction costs shown in the bottom row of Panels A and B in Table 1. At a lower base price of \$50, monitoring is preferred because the total savings are 0.018%. In contrast, at a higher base price of \$100, it is better not to monitor because the savings are -0.002%.

The key to the model is the difference in the nature of the two components of (dollar) execution costs – (dollar) adverse selection and (dollar) discreteness related commissions. The former increases in proportion to the base price whereas the latter, being fixed, stays the same at all price levels. Therefore, discretionary liquidity are indifferent about the price level with respect to the savings in adverse selection commissions (0.978% at both base prices). However, they do care about the price level with respect to savings in discreteness related commissions (0.02% at the higher base price of \$100, but 0.04% at the lower base price of \$50).

At the lower base price of \$50, the savings in discreteness related commissions are sufficiently high, and total savings in execution costs (adverse selection and discreteness related commissions) offset monitoring costs. Monitoring the market is therefore beneficial to liquidity traders with low monitoring costs. In contrast, at the higher base price of \$100, monitoring is *not* beneficial. Hence, liquidity traders with low monitoring costs endogenously choose to act as discretionary traders when the base price is \$50, but prefer to act as nondiscretionary traders when the base price is \$100. As a result, when the base price is \$50, the trading pattern across the two periods is either (100, 20) or (20, 100). In contrast, when the base price is \$100, the trading pattern is (60, 60). Thus, the base price level affects the distribution of liquidity traders across the two periods.

Panel C in Table 1 shows the total transaction costs due to adverse selection, discreteness, and monitoring incurred by all liquidity traders at the two base prices. For the computations in Panel C of Table 1, we assume that the adverse selection commission is \$0.575 when the number of liquidity traders in a period is 20. This situation arises in one of the periods when the base price is \$50. To read Panel C in Table 1, consider the first row where the base price is \$50. 100 liquidity traders face an adverse selection commission of \$0.046 and 20 liquidity traders face an adverse selection commission of \$0.575. On a per dollar basis, the total adverse selection commissions are $[100 \times \$0.046 + 20 \times \$0.575]/\$50 = 0.322$. We refer to this sum of all adverse selection commissions as the adverse selection *component* of total transaction costs.

Furthermore, 100 liquidity traders face discreteness related commissions of \$0.04 and 20 liquidity traders face discreteness related commissions of \$0.08 (this is less than \$0.10 because they may be just lucky and trade in a period with discreteness related commissions of \$0.02). The total discreteness related commissions on a per dollar basis is $[100 \times \$0.04 + 20 \times \$0.08]/\$50 = 0.112$ (we refer to the sum of all discreteness related commissions as the discreteness related *component* of total transaction costs).

Finally, 80 liquidity traders incur monitoring costs of 1%, implying total monitoring costs of $[80 \times (0.01 \times \$50)/\$50] = 0.80$ on a per dollar basis. This is the monitoring cost *component* of total transaction costs. The total transaction costs are $[0.322 + 0.112 + 0.80] = 1.234$ on a per dollar basis. Note that this is the total transaction cost of all liquidity traders, taken together as a group.

In contrast, when the base price is \$100, the total transaction costs (on a per dollar basis) are 1.356. From the firm's perspective, the lower base price of \$50 is preferable because liquidity traders (nondiscretionary and discretionary, taken together as a group) face lower total transaction costs on a per dollar basis.

Panel C in Table 1 also shows that the adverse selection component is increasing in the base price. This situation arises because a lower base price is associated with more concentrated trading. Consequently, many liquidity traders incur low adverse selection commissions, resulting in a lower adverse selection component. In contrast, the discreteness related and the monitoring cost components are decreasing in the base price. This opposite relationship provides the tradeoffs for an optimal price level.

In contrast to the numerical example, the model allows for a continuum of monitoring costs for liquidity traders, a continuum of discreteness related commissions, a continuum of base prices, and multiple (although, finite) rounds of trading opportunities. More importantly, the adverse selection and discreteness related commissions are endogenously determined.

The intuition of the model can also be explained as follows. A lower base price induces more liquidity traders to act as discretionary traders. This is beneficial because greater discretionary trading results in a lower adverse selection component. However, a lower base price also has adverse cost implications. First, the discreteness related commission (*DRC*) component increases and higher (cumulative) monitoring costs are incurred because more liquidity traders act as discretionary traders. The optimal price, which results in an optimal amount of discretionary trading, is the one equating the marginal adverse selection component on the one hand to the sum of the marginal *DRC* and the marginal monitoring cost component on the other hand.

3. The model

This section develops a market microstructure model that captures the role of the asset price level in determining the behavior of market participants. The asset price process is given by $P_\tau = P_0 + \sum_{t=1}^{\tau} \delta_t$, where P_τ is the underlying asset price at time τ, P_0 is an initial base price and $\delta_t[\equiv N(0, \sigma^2)]$ represents an unanticipated piece of (short-lived) private information that is revealed at the end of each period t.

We also assume that σ is linear in the base price, i.e., $\sigma(P_0) = kP_0$, where k is referred to as the volatility parameter.[7] This characterization recognizes that the magnitude of private information released in each period is proportional to the underlying asset value. The rest of the economy is characterized by the following assumptions:

(A1) The size of the trading population is T and there are m trading periods.

(A2) Risk neutral market makers post competitive prices before accepting order flow. Market makers do not incur order processing costs and do not face any inventory constraints.

(A3) A fraction $(1 - \lambda)$ of the trading population (T) consists of cash constrained risk neutral informed traders who trade on short-lived information in each one of the m periods. They obtain (identical) perfect signals of δ_t at the beginning of *each* period t.

(A4) A fraction λ of the trading population (T) consists of risk neutral uninformed liquidity traders.

A2 ensures that market makers post ask and bid prices such that the expected losses to informed traders are offset by the expected profits from uninformed liquidity traders (as in Admati and Pfleiderer, 1989). A3 implies that informed traders cannot assume unbounded positions to take advantage of the perfect signal because of wealth constraints (again, as in Admati and Pfleiderer, 1989). Their order size is normalized to 1 for convenience. Note, δ is short-lived information that is revealed at the end of each period. Therefore, in order to utilize their (exogenously) acquired private information, informed traders must trade in the same period they receive information. For convenience, we assume that in each period, $t \in (1, m)$, the same informed traders are observing a private signal (δ_t) and taking positions based upon this information.

[7] Our assumption of linearity is consistent with the standard assumption in asset pricing literature. It mathematically follows that splitting an asset into n equal parts results in the standard deviation of each part being equal to $(1/n)$th the standard deviation of the original asset. In other words, standard deviation is linearly related to underlying asset value.

3.1. Equilibrium commissions

Consider the ask side of the market (the analysis is identical for the bid side of the market). For the competitive, risk neutral market maker, the equilibrium ask commission (a^*) can be determined by setting his expected profits to zero. Given A3, the number of informed traders in each period is $(1-\lambda)T$. For purposes of illustration let the remaining uninformed liquidity traders (λT) be equally distributed across the m periods. Then, we get the equilibrium commission (a^*) by solving the following equation (see Appendix A for the derivation):

$$-(1-\lambda)T\left\{\sigma(P_0)\phi\left(\frac{a}{P_0}\right)-\left[1-\Phi\left(\frac{a}{P_0}\right)\right]a\right\}+\left(\frac{\lambda T}{m}\right)a=0, \tag{1}$$

where $\phi(.)$ and $\Phi(.)$ represent the probability density function and the cumulative distribution function of the standard normal distribution, respectively. The left hand side of Eq. (1) shows the expected profits of the market makers, which is made up of two components – the first term represents the expected losses to informed traders and the second term represents the expected profits from liquidity traders.

Note that T factors out of Eq. (1). Thus, the trading population (T) is irrelevant for the analysis. Also, if a^* is the solution to Eq. (1), then, under continuous prices, the ask price (A_c) is equal to $P_{\tau-1}+a^*$. We refer to a^* as the adverse selection commission. Because $\sigma(P_0)$ increases linearly in P_0, it turns out that the (dollar) adverse selection commission (a^*) also increases linearly in P_0. However, as shown in Appendix A the adverse selection commission *per dollar traded* (i.e., percentage commissions) is constant and independent of the base price (P_0).

3.2. Discreteness related commissions (DRC)

Under discrete prices (separated by ticks of size d), the market maker's pricing policy is different. In all likelihood, it may not be feasible to set the price at $A_c=P_{\tau-1}+a^*$ because A_c may not be an exact multiple of the tick size (d). Anshuman and Kalay (1998) show that, under discrete prices, competitive market makers round the ask price upward to the nearest feasible price (similarly, on the bid side of the market, the continuous-case bid price is rounded downward to the nearest feasible price).[8] Therefore, the discreteness

[8] Anshuman and Kalay (1998) examine the impact of discrete pricing restrictions in greater detail. Following them, we assume that there can be no cross-subsidization of profits across time, i.e., market makers could sell below a^* in one period and sell above a^* in the other period, thereby, selling at an average commission of a^*. Such a linear combination of trades, i.e., splitting orders and executing them at adjacent prices, is assumed to be very costly. Alternatively, one can assume that the market maker is not allowed to use mixed strategies in his pricing rule.

related commissions on the ask side of the market are equal to $d - \text{Mod}[P_0 + a^*, d]$ (if $\text{Mod}[P_0 + a^*, d] > 0$) or equal to 0 (if $\text{Mod}[P_0 + a^*, d] = 0$).

The restriction on discreteness of prices results in a few interesting implications. First, due to discreteness, there is an additional component of transaction costs, henceforth referred to as *DRC*. The equilibrium commission is going to vary in the range $[a^*, a^* + d)$, depending on the location of A_c $(= P_{\tau-1} + a^*)$ on the real line. Second, given that $P_{\tau-1}$ and a^* are common knowledge at time τ, all market participants can infer the exact magnitude of *DRC* in the current period.

3.3. Strategic liquidity trading

Liquidity traders can reduce transaction costs by deferring their trades to a period where *DRC* are very low. More importantly, they would also face lower adverse selection commissions because of the ensuing concentration of trades. The benefits of strategically timing trades can be a significant reduction in execution costs.

Of course, such strategic behavior is not costless. It involves close monitoring of the market to take advantage of periods with low *DRC*. The monitoring costs for a liquidity trader depends on the opportunity cost of his or her time. From here on, we recognize that liquidity traders face differential opportunity costs of monitoring.

(A5) At time $t = -\infty$, risk neutral liquidity traders (λT) make a strategic decision – whether to act as discretionary or nondiscretionary traders. This decison depends on their personal opportunity costs of monitoring. We assume that, on a continuum of increasing monitoring costs, the qth percentile liquidity trader incurs a (per dollar) monitoring cost, $C(q) = f/[-\ln(q)]^{1/w}$, where $f > 0$ and $w > 1$.

At time $t = -\infty$, all liquidity traders are potential discretionary traders. Liquidity traders weigh the benefits of discretionary trading (namely, lower execution costs) against their personal opportunity costs of monitoring. Only those liquidity traders who foresee a net benefit choose to act as discretionary traders. We assume that, on a continuum of increasing monitoring costs, the qth percentile liquidity trader incurs a (per dollar) monitoring cost, $C(q) = f/[-\ln(q)]^{1/w}$, where $f > 0$ and $w > 1$.[9] In general, the parameter f tends to "shift" $C(q)$ up or down and the parameter w tends to alter the shape of the function $C(q)$. This is illustrated in Fig. 1, which shows the cost function for a

[9] By constraining f and w to be greater than 0, we ensure that $C'(q) > 0$. Thus, $C(q)$, which represents the personal monitoring cost incurred by the qth percentile liquidity trader, is increasing in q, by construction. Note that $C(0) \to 0$ and $C(1) \to \infty$. Therefore, traders differ in monitoring costs over the interval $(0, \infty)$. The constraint $w > 1$ is required for proper integration of the cost function, as discussed in Appendix C.

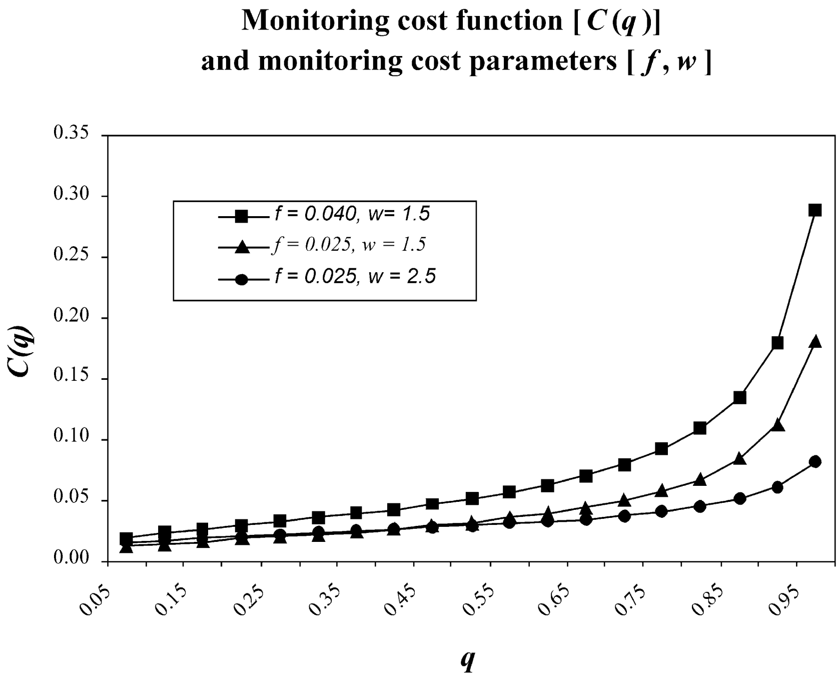

Fig. 1. The monitoring cost function. The monitoring cost function specifies the opportunity cost of monitoring incurred by the qth percentile trader. The cost function is $C(q) = f/[-\ln(q)]^{1/w}$, where $f > 0$ and $w > 1$. The monitoring cost parameters f and w affect the shape of the cost function, as shown in the three representative situations in the graph. In general, the parameter f tends to "shift" the cost function up or down, whereas the parameter w tends to affect the curvature of the cost function.

few combinations of the parameter values f and w. We refer to f and w as the monitoring cost parameters.

If the q^* percentile liquidity trader's personal monitoring cost just offsets the savings in execution costs from timing trades, he would be indifferent between acting as a discretionary or a nondiscretionary trader. Assuming that he chooses to act as a discretionary trader, the fraction of λT liquidity traders who act as discretionary traders is q^*, in equilibrium. The remaining fraction $(1 - q^*)$ would rationally choose to act as nondiscretionary traders because they face higher monitoring costs than that of the q^* percentile liquidity trader.

(A6) All liquidity traders realize their trading requirements at time $t = 0^-$. Discretionary liquidity traders can trade in any one of the m periods. Waiting costs are negligible and the arrival rate of nondiscretionary liquidity traders into the market is constant.

Recall, the total trading population is T. Among these, a fraction $(1 - \lambda)T$ are informed traders who trade in each one of the m periods. The remaining fraction λT consists of liquidity traders. Among the liquidity traders, a fraction

$q^*\lambda T$ choose to act as discretionary traders and aggregate their trades in one of m periods (with low DRC). The remaining fraction $(1-q^*)\lambda T$ consists of nondiscretionary traders.[10] Given that there are negligible waiting costs,[11] nondiscretionary traders are indifferent between trading early or late. We assume that they arrive in the market at a constant rate.[12] In other words, nondiscretionary traders are distributed equally across all the m periods.

The trading pattern consists of a single period of concentrated trading and $m-1$ periods of "regular" trading. Let T_D represent the number of discretionary traders and T_ND represent the number of nondiscretionary traders per period. Then,

$$T_\mathrm{D} = q^*(\lambda T), \tag{2}$$

$$T_\mathrm{ND} = (1-q^*)(\lambda T/m). \tag{3}$$

In the period of concentrated trading, DRC and adverse selection commissions are low compared to the remaining periods. Let the adverse selection commission in the period of concentrated trading be a_l and let the adverse selection commission in the remaining periods be a_h. Note, a_l is less than a_h because of the presence of additional liquidity traders in the period of concentrated trading. The equilibrium adverse selection commissions, a_l and a_h, are given by the solutions of Eqs. (4) and (5), respectively, where the market maker's expected profit function is set to zero. These equations are identical to Eq. (1), except that the number of liquidity traders is different:

$$a_\mathrm{l}\colon -(1-\lambda)T\left\lfloor \sigma(P_0)\phi\left(\frac{a}{P_0}\right) - \left[1-\Phi\left(\frac{a}{P_0}\right)\right]a\right\rfloor + (T_\mathrm{d}+T_\mathrm{ND})a = 0, \tag{4}$$

$$a_\mathrm{h}\colon -(1-\lambda)T\left\lfloor \sigma(P_0)\phi\left(\frac{a}{P_0}\right) - \left[1-\Phi\left(\frac{a}{P_0}\right)\right]a\right\rfloor + (T_\mathrm{ND})a = 0. \tag{5}$$

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[10] We refer to discretionary traders who do not exercise their flexibility as nondiscretionary traders. In our model, nondiscretionary liquidity traders realize their trading requirements before the market opens at time $t=0^-$. This differs from the traditional view in the market microstructure literature, where nondiscretionary liquidity traders realize their trading requirements in a particular period after the market opens and are compelled to trade in the same period.

[11] The assumption of negligible waiting costs is reasonable in an intraday trading scenario where the trading horizon is of the order of a few hours, at most. Essentially, we assume that a zero discount rate applies over the trading horizon.

[12] Alternative assumptions about the arrival rate of nondiscretionary liquidity traders would imply exogenously imposed excess liquidity trading in at least one period. The model can be suitably altered to accommodate any given specification of nondiscretionary liquidity trader behavior. However, we believe that there is no ex-ante motivation to justify examining alternative specifications. Assuming a uniform arrival rate of nondiscretionary traders seems to be the most innocuous specification.

Since discretionary traders have to monitor the market from the very first period, the decision to act as a discretionary trader or nondiscretionary trader is made before the market opens. Hence q^*, and therefore, a_l and a_h are completely determined before trading begins.

By pooling their trades in any chosen period, discretionary traders can save $(a_h - a_l)$ on adverse selection commissions. The savings in adverse selection commissions would be the same no matter which period they choose to aggregate their trades. However, in a world with discrete prices, the savings in *DRC* are subject to timing ability because *DRC* are time varying. Hence timing matters. The only uncertainty is with respect to the realization of *DRC* over the interval $(0, d)$.

3.3.1. Discretionary traders' timing strategy

As discussed in Section 3.2, current period *DRC*, is common knowledge at the beginning of each period, but future period *DRC* are uncertain. Being risk neutral, discretionary traders weigh the current period *DRC* with the expected *DRC* upon deferring trades. Thus, the distribution of *DRC* in future periods affects the timing strategy of discretionary traders.

Suppose *DRC* are uniformly distributed over $(0, d)$. Consider a trading horizon (m) of two periods. At the beginning of the first period, *DRC* for the first period are known, but *DRC* for the second (and last) period are unknown. Risk neutral discretionary traders can compare the current realized *DRC* with the expected *DRC* upon deferring trades, which are equal to $d/2$. If the current *DRC* are less than or equal to $d/2$, it makes sense to trade immediately. In contrast, if the current $DRC > d/2$, it makes sense to defer trades to the second period. Thus, the timing strategy involves a simple trading rule. In the first period (of a two period horizon), the trading rule would be to trade in the current period if $DRC \leqslant d/2$, otherwise to defer trades. We refer to the fraction $\frac{1}{2}$ in $d/2$ as the cutoff level that describes the trading behavior of discretionary traders in the first period. Note that the cutoff level indicates the expected *DRC* from deferring trades.

In general (over an m-period trading horizon), the timing strategy would involve a trading rule that employs a critical cutoff (expressed as a fraction of the tick size) corresponding to each period. If the realized *DRC* is less than or equal to that implied by the cutoff level (relevant for that period), discretionary traders are better off trading in that period, as opposed to deferring trades. Conversely, if the realized *DRC* is larger than that implied by the cutoff level, it is better to defer trades to the next period.

Note that the cutoff for the last period has to be equal to 1, because discretionary traders are forced to trade in this period (if they have deferred trades until then). In conclusion, the trading rule therefore implies that discretionary traders should trade in the first period that has a realized *DRC* less than or equal to the cutoff level (corresponding to that period). Such a trading rule ensures minimization of expected *DRC*.[13]

Proposition 1. The timing strategy of discretionary traders can be described by a set of optimal cutoffs $((\alpha_t^*,\ 0<\alpha_t^*\leqslant 1))$, *that is determined by recursively solving* Eq. (6) *from* $t=(m-1),\ \ldots,\ 1$, *using the end-game constraint* $\alpha_m^*=1$. *Discretionary traders would find it optimal to trade in the first period that has a realized DRC less than or equal to* $\alpha_t^* d$.

$$\begin{aligned}\alpha_t^* = z_t : z_t d = {} & F_{t+1}(\alpha_{t+1}^* \mid DRC_t = z_t d) E\{z_{t+1} d \mid 0 \leqslant z_{t+1} \leqslant \alpha_{t+1}^*, DRC_t = z_t d\} \\ & + (1 - F_{t+1}(\alpha_{t+1}^* \mid DRC_t = z_t d)) \\ & \quad F_{t+2}(\alpha_{t+2}^* \mid DRC_t = z_t d) E\{z_{t+2} d \mid 0 \leqslant z_{t+2} \leqslant \alpha_{t+2}^*, DRC_t = z_t d\} + \cdots \\ & + (1 - F_{t+1}(\alpha_{t+1}^* \mid DRC_t = z_t d)) \cdots (1 - F_{m-1}(\alpha_{m-1}^* \mid DRC_t = z_t d)) \\ & \quad F_m(\alpha_m^* \mid DRC_{m-1} = z_{m-1} d) E\{z_m d \mid 0 \leqslant z_m \leqslant \alpha_m^*, DRC_t = z_t d\}, \end{aligned} \tag{6}$$

where the realization DRC_t *in period t is equal to* $z_t d$, $0\leqslant z_t<1$ and $F_t(.|DRC_{t-1})$ *refers to the cumulative distribution function of the conditional distribution of* DRC_t *given* DRC_{t-1}).

Proof. Appendix B. □

Proposition 1 describes the timing strategy of discretionary traders. In deciding whether to trade in the current period or to defer trading to the next period, discretionary traders compare the current period realized *DRC* with the expected *DRC* upon deferring trades. Eq. (6) presents this comparison at stage *t* of the trading horizon of *m* periods. Note that the distribution of future period *DRC* depends on the realized *DRC* in the current period. Hence Eq. (6) deals with the conditional distribution of *DRC*. The optimal cutoffs can be determined by solving Eq. (6) using a recursive backward dynamic programming approach, where an end-game constraint ($\alpha_m^*=1$) applies.

The trading rule works as follows: If the realization of DRC_1 in Period $1\leqslant\alpha_1^* d$, then discretionary traders would trade in Period 1, otherwise they would defer their trades to the next period. Suppose discretionary traders prefer to defer their trades and reach Period 2. If the realization of DRC_2 in Period $2\leqslant\alpha_2^* d$, then discretionary traders would trade in the Period 2, otherwise they would defer their trades to the next period, and so on till they

[13] Note that discretionary traders would be interested in minimizing the expected execution costs of $(a_l + DRC)$. It turns out that minimizing expected *DRC* also ensures that a_l would be minimized. This follows because q^* is increasing in the savings in execution costs, $S(\alpha_1^*, \ldots, \alpha_m^*, q^*)$, as discussed later in Eq. (11). Furthermore, as shown in Eq. (10), $S(\alpha_1^*, \ldots, \alpha_m^*, q^*)$ is inversely related to discretionary trader's expected *DRC*, $\mathrm{E}(DRC)_\mathrm{D}$. Finally, since a_l is monotonically decreasing in q^* [see Eqs. (4) and (5)], it follows that minimizing expected *DRC* ensures that a_l is also minimized.

reach a period with *DRC* less than or equal to the cutoff relevant for that period.

3.3.2. Ex-ante expected execution costs of discretionary traders

As stated in Assumption A5, liquidity traders make a strategic decision on whether to act as discretionary traders or not, at time $t = -\infty$. At this point in time, the distribution of *DRC* is Uniform (0, *d*) because there is no information available about the price process.[14] Knowing the cutoffs $\alpha_1^*, \ldots, \alpha_m^*$, one can compute the ex-ante (at time $t = -\infty$) expected *DRC* incurred by discretionary traders [E$(DRC)_\mathrm{D}$]. Therefore,

$$\mathrm{E}(DRC)_\mathrm{D} = \alpha_1^*(\alpha_1^* d/2) + (1-\alpha_1^*)\alpha_2^*(\alpha_2^* d/2) + \cdots + (1-\alpha_1^*)(1-\alpha_2^*)\cdots(1-\alpha_{m-1}^*)\alpha_m^*(\alpha_m^* d/2). \tag{7}$$

Given that discretionary traders incur adverse selection commissions (a_l, which depends on q^*), the expected per dollar execution costs of discretionary liquidity traders (EC_D) is given by

$$EC_\mathrm{D}(\alpha_1^*, \ldots, \alpha_m^*, q^*) = [a_\mathrm{l} + E(DRC)_\mathrm{D}]/P_0. \tag{8}$$

3.3.3. Equilibrium amount of discretionary trading (q^*)

To determine the equilibrium amount of discretionary traders (q^*), we first determine the savings in execution costs due to timing of trades. The nondiscretionary traders who trade in $(m-1)$ regular periods expect to pay an adverse selection commission of a_h and, on average, $d/2$ in *DRC*.[15] However, if

[14] As discussed in Appendix B, the distribution of *DRC* is given by the wrapped normal distribution. Mardia (1972) shows that the wrapped normal distribution converges to the uniform distribution when $\rho = \exp[(-1/2)\sigma^2]$ tends to zero, where σ^2 is the variance of the underlying normal distribution. At time $t = -\infty$, the relevant underlying normal variable is $\Sigma\delta$ over the time interval $(-\infty, 0)$, whose variance approaches infinity. Hence the distribution of *DRC* in Period 1 through Period *m* will be uniform because the wrapped normal distribution converges to the uniform distribution.

[15] It might seem that *DRC* in the regular periods should vary over the interval $(\alpha_t^* d, d)$, otherwise discretionary traders would pool their trades in such periods. However, this inference is incorrect. Note *DRC* depend on the location of the continuous-case ask price $A_\mathrm{c}(= P_{\tau-1} + a^*)$ on the real line. Given $P_{\tau-1}$, the location of A_c depends on the equilibrium commission (a^*), which depends on the number of liquidity traders trading in a period. If discretionary traders are trading, the appropriate continuous-case ask price is given by $A_\mathrm{c} = P_{\tau-1} + a_\mathrm{l}$, whereas when only nondiscretionary traders appear in the market, the continuous-case ask price is given by $A_\mathrm{c} = P_{\tau-1} + a_\mathrm{h}$. Hence, discretionary traders defer their trades whenever, conditional on their trading, the continuous-case ask price (by $A_\mathrm{c} = P_{\tau-1} + a_\mathrm{l}$) is such that *DRC* lie in the interval $(\alpha_t^* d, d)$. Only nondiscretionary traders would then trade, and it is quite possible that *DRC* are less than $\alpha_t^* d$ because the continuous-case ask price would then be given by $A_\mathrm{c} = P_{\tau-1} + a_\mathrm{h}$. However, discretionary traders cannot take advantage of this situation because if they trade, *DRC* would lie in the interval $(\alpha_t^* d, d)$. In general, *DRC* in a regular period, where only nondiscretionary traders trade, would vary over $(0, d)$.

they are lucky and realize their trading need in the period of concentrated trading, their expected execution costs are equal to $\{a_l + \mathrm{E}(DRC)_D\}$, the same as that of discretionary liquidity traders. Ex-ante, the probability of trading in a regular period is $(m-1)/m$ and the probability of trading in the period of concentrated trading is $(1/m)$. Thus, the expected (per dollar) execution costs incurred by a nondiscretionary trader is given by

$$\begin{aligned} EC_{ND}(\alpha_1^*, \ldots, \alpha_m^*, q^*) = {} & [(m-1)/m](a_h + d/2)/P_0 \\ & + (1/m)[a_l + \mathrm{E}(DRC)_D]/P_0. \end{aligned} \tag{9}$$

It follows that the (per dollar) savings in executions costs due to timing of trades is given by $S(\alpha_1^*, \ldots, \alpha_m^*, q^*) = EC_{ND}(\alpha_1^*, \ldots, \alpha_m^*, q^*) - EC_D(\alpha_1^*, \ldots, \alpha_m^*, q^*)$, as stated in

$$S(\alpha_1^*, \ldots, \alpha_m^*, q^*) = [(m-1)/m][(a_h - a_l) + d/2 - \mathrm{E}(DRC)_D]/P_0. \tag{10}$$

Liquidity traders compare the savings in execution costs to their personal monitoring costs. In equilibrium, q^* would be such that the q^* percentile liquidity trader would be indifferent between acting as a discretionary or as a nondiscretionary trader. In either case, he would incur identical total expected (per dollar) transaction costs. In short, q^* would be such that savings in execution cost from timing trades would be exactly offset by his personal monitoring costs. Thus, $C(q^*) = S(\alpha_1^*, \ldots, \alpha_m^*, q^*)$. Plugging the functional form of $C(q)$, as defined in A5, we get

$$q^* = \exp[-\{f/S(\alpha_1^*, \ldots, \alpha_m^*, q^*)\}^w]. \tag{11}$$

3.3.4. Equilibrium solution

Fig. 2 shows the pattern of liquidity trading in the m-period economy. In the figure, the period of concentrated trading is Period s. Ex-ante, however, the period of concentrated trading is unknown. Given a base price (P_0), discretionary traders choose the optimal cutoffs, $\alpha_1^*, \ldots, \alpha_{m-1}^*$, by recursively solving Eq. (6). The period of concentrated trading depends on the realization of DRC in the m periods. The first period that has a realized DRC less than or equal to the relevant cutoff for that period will be the period of concentrated trading.

The optimal cutoffs determine the ex-ante expected DRC incurred by discretionary traders, as described in Eq. (7) and the discretionary traders' savings in execution costs, as described in Eq. (10). The equilibrium amount of concentrated trading (q^*) is then solved for, as shown in Eq. (11). Knowing q^*, the adverse selection commissions, a_l, and a_h, are found using Eqs. (4) and (5), respectively. The solution set is given by $\{\alpha_1^*, \ldots, \alpha_m^*, q^*, a_h, a_l, P_0\}$, which corresponds to a given base price, P_0.

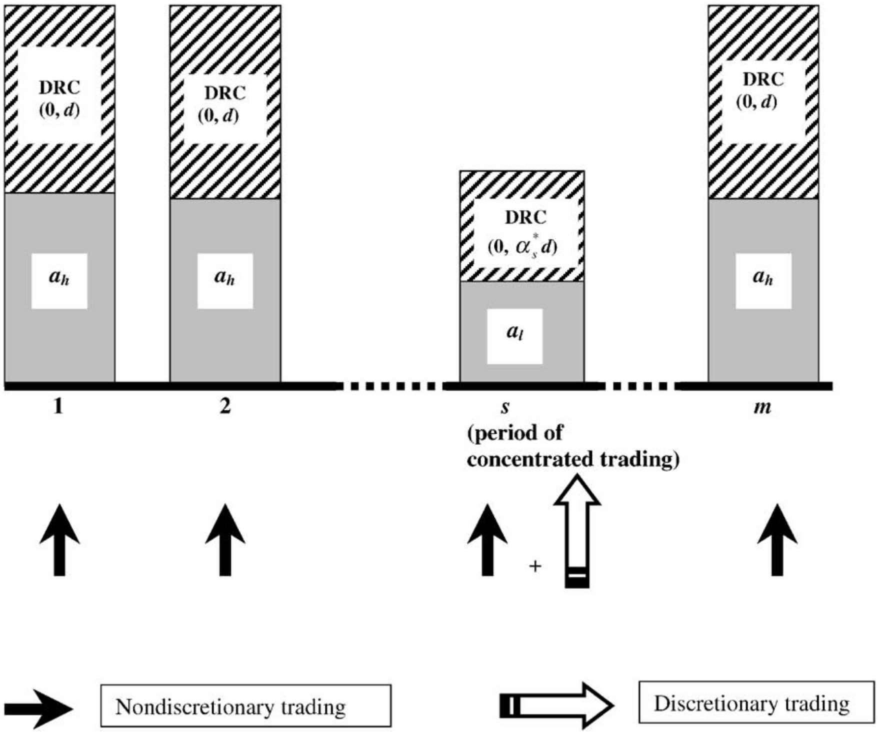

Fig. 2. A pictorial representation of the execution costs. The economy consists of m trading periods. A fraction q^* (endogenously determined) of the total number of liquidity traders (λT) choose to act as discretionary traders and the remaining fraction $(1-q^*)$ prefer to act as nondiscretionary traders. Before the market opens, discretionary traders choose optimal cutoff level α_t^* $(0<\alpha_t^*<1)$ corresponding to each period t. They trade in the first period that has discreteness related commissions $(DRC)<\alpha_t^* d$, where d is the tick size. If the period of concentrated trading occurs in Period s, DRC vary over $(0, \alpha_s^* d)$ and the adverse selection related commission is equal to a_l. In the remaining $(m-1)$ regular periods, the adverse selection related commission is equal to a_h and DRC varies over the interval $(0, d)$.

3.4. Equilibrium transaction costs

We define the per dollar total transaction, $TC(P_0)$, as follows:

$$TC(P_0) = (m-1)\left[\frac{a_h + d/2}{P_0}\right] T_{ND} + \left[\frac{a_l + E(DRC)_D}{P_0}\right](T_{ND} + T_D) + \int_0^{q^*} \lambda\, TC(q)\, dq. \tag{12}$$

In the above equation, the first term within the square brackets represents the *expected* (per dollar) execution costs incurred by nondiscretionary traders in $(m-1)$ periods of regular trading, given that DRC are, on average, equal to $d/2$. We explicitly factor in the depth of the market by including the number of nondiscretionary traders (T_{ND}) incurring these commissions. Similarly, the second term reflects the *expected* (per dollar) execution costs incurred by liquidity traders in the period of concentrated trading. Finally, the last term shows the cumulative monitoring costs incurred by all liquidity traders who act as discretionary traders. It can be shown (see Appendix C) that the integration of $\lambda\, TC(q)$ over the interval $(0, q^*)$ gives the expression: $f\lambda T\Gamma((w-1)/w))$ $\{1 - GAMMADIST[(-\mathrm{Ln}(q^*)]\}$, where $\Gamma(.)$ is the gamma function and $GAMMADIST$ is the cumulative distribution function of the standard gamma distribution with parameter $[(w-1)/w]$.

The expressions in the square bracket are normalized by the base price (P_0). This normalization is required to remove any spurious price effects. Thus, our objective function is expressed on a per dollar basis. This (inverse) measure of liquidity reflects both the spread (i.e., commission) and the depth in the market. Note, in Eq. (12) the base price appears explicitly in the denominator, and implicitly in a_{h}, a_{l}, T_{D}, T_{ND} (through q^* which depends on P_0).

4. Numerical solution of the model

The model cannot be solved in closed-form. Therefore, we numerically solve the model for reasonable parameter values. The numerical solution set $\{\alpha_1^*, \ldots, \alpha_m^*, q^*, a_{\mathrm{h}}, a_{\mathrm{l}}; P_0\}$ is used to compute the value of the transaction cost function $TC(P_0)$. Repeating this exercise for different values of P_0 generates the functional form of $TC(P_0)$. The optimal base price is the one that results in the lowest transaction cost.

4.1. The optimal cutoffs

The first step in the numerical solution procedure is to solve Eq. (6) to determine the optimal cutoffs, $\alpha_1^*, \ldots, \alpha_m^*$. For these computations, we let the number of trading periods (m) equal 10, the volatility parameter (k) equal 0.02, and the tick size equal \$0.125. A value of $k = 0.02$ implies a standard deviation of 2% (of the price level), which is consistent with observed daily standard deviations.[16] Appendix B develops the functional form of the conditional distribution, $F_t(\alpha_t^* \mid DRC_{t-1} = z_{t-1}d)$, and the expectation, $E\{z_t d \mid z_t \leqslant \alpha_t^*, DRC_t = z_t d\}$. These terms appear in Eq. (6). Table 2 shows the optimal cutoffs at different base prices varying from \$1/2 to \$100. The optimal cutoffs

[16] Typical values of volatility of stocks lie in the range of 20–40% per annum, or equivalently 1.046–2.093% on a daily basis. Thus our choice of the parameter value is consistent with the daily standard deviations observed on stock exchanges.

Table 2
The optimal cutoffs

This table shows the optimal cutoffs (α_i^*) and discretionary traders' ex-ante expected discreteness related commissions [$E(DRC)_D$] using a dynamic optimization procedure. The problem has been solved for $m=10$ periods for different base prices (P_0). The base price level affects the standard deviation of the private information ($\sigma = kP_0$), where k is the volatility parameter. The m distinct cutoffs (expressed as a fraction of the tick size) appear in the rows. We assume that the volatility parameter (k) is equal to 0.02 and the tick size (d) is equal to \$0.125.

Cutoff (α_i^*)	Base price (P_0)					
	\$1/2	\$1	\$2	\$10	\$50	\$100
α_1^*	0.3839	0.1536	0.1492	0.1502	0.1502	0.1499
α_2^*	0.4020	0.1763	0.1629	0.1636	0.1635	0.1631
α_3^*	0.4176	0.2099	0.1798	0.1797	0.1797	0.1792
α_4^*	0.4318	0.2629	0.2011	0.1996	0.1996	0.1992
α_5^*	0.4450	0.3281	0.2291	0.2249	0.2249	0.2246
α_6^*	0.4579	0.3842	0.2675	0.2583	0.2583	0.2581
α_7^*	0.4708	0.4285	0.3224	0.3047	0.3047	0.3046
α_8^*	0.4843	0.4653	0.4000	0.3750	0.3750	0.3750
α_9^*	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
α_{10}^*	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$E(DRC)_D$	0.0256	0.0183	0.0174	0.0174	0.0174	0.0174

depend on the base price because σ^2 (the variance of δ) depends on the base price ($\sigma = kP_0$).

Consider the case when the base price is \$50. The cutoff for the first period is 0.1502. This means that discretionary traders would find it optimal to trade in the first period only if the realized DRC in Period 1 is less than or equal to $0.1502d=\$0.01877$ (assuming a tick size of \$0.125). Otherwise, they would defer their trades to the next period. The last period cutoff is always 1, since discretionary are constrained to trade within the trading horizon ($m=10$ periods). It turns out that the optimal cutoffs are not very sensitive to the base price, except at the very low base price of about \$1.

Next, we apply the optimal cutoffs to Eq. (7) and determine discretionary traders' ex-ante expected DRC at each base price. This computation appears in the bottom row of Table 2. For a base price of \$50, the $E(DRC)_D=\$0.0174$, which is significantly lower than the nondiscretionary trader's expected DRC of \$0.0625.

4.2. The transaction cost function, $TC(P_0)$

To construct the transaction cost function, we must first solve for the remaining endogenous variables in the solution set, namely, q^*, a_h, and a_l, corresponding to each base price level (P_0). For convenience, we assume that

Table 3
Equilibrium characteristics

This table shows the numerical solution of the model at different base prices. $P_0 \equiv$ base price, $q^* \equiv$ fraction of liquidity traders who choose to act as discretionary traders, $a_h \equiv$ adverse selection related commissions in a regular period, $a_l \equiv$ adverse selection related commissions in the period of concentrated trading, $T_D \equiv$ number of discretionary traders, and $T_{ND} \equiv$ number of nondiscretionary traders in each period. We assume that, in a continuum of increasing costs, the qth percentile liquidity traders faces a monitoring cost, $C(q) = f/[-\ln(q)]^{1/w}$, where $f > 0$ and $w > 1$. The parameters defining the numerical solution are as follows: (i) λ: the fraction of liquidity traders in the trading population (T), (ii) k: the volatility parameter, which specifies the standard deviation of the private information (δ) in $\sigma(P_0) = kP_0$, where P_0 is the base price, (iii) m: the number of periods, (iv) d: the tick size, and (v) (f, w): the monitoring cost parameter pair that defines the monitoring cost schedule. The parameters chosen for the simulation are (i) $\lambda = 0.6$, $T = 200$, (ii) $k = 0.02$, (iii) $m = 10$, (iv) $d = \$0.125$, and (v) $f = 0.0109$, $w = 1.6$.

P_0	a_h/P_0	a_l/P_0	q^*	T_D	T_{ND}
0.5	0.0398	0.0042	0.9709	116.51	0.35
1	0.0362	0.0042	0.9486	113.83	0.62
2	0.0319	0.0044	0.9020	108.25	1.18
10	0.0247	0.0051	0.7169	86.03	3.40
20	0.0227	0.0055	0.6258	75.10	4.49
30	0.0218	0.0058	0.5745	68.94	5.11
40	0.0213	0.0061	0.5389	64.67	5.53
50	0.0209	0.0062	0.5115	61.37	5.86
52	0.0208	0.0063	0.5066	60.79	5.92
53	0.0208	0.0063	0.5042	60.50	5.95
54	0.0208	0.0063	0.5019	60.22	5.98
60	0.0206	0.0064	0.4885	58.62	6.14
70	0.0203	0.0066	0.4680	56.16	6.38
80	0.0200	0.0067	0.4485	53.82	6.62
90	0.0198	0.0069	0.4283	51.39	6.86
100	0.0189	0.0077	0.3453	41.43	7.86

the total trading population (T) is equal to 200 (the results are invariant to the choice of T) and that the fraction of liquidity traders (λ) is equal to 60% or 0.6. The other key parameters are the monitoring cost parameters (f and w). They define the shape of the monitoring cost schedule faced by liquidity traders. We solve for q^*, a_h, and a_l, at different base prices (P_0) using the following parameter values: $d = \$0.125$, $k = 2\%$, $m = 10$, $T = 200$, $\lambda = 0.6$, and $(f, w) \equiv (0.0109, 1.6)$. We find that the resulting transaction cost function, $TC(P_0)$, exhibits a local interior minimum at a base price of \$53.

Table 3 presents the numerical solutions. Given $\lambda = 0.6$, the number of uninformed liquidity traders (λT) is equal to 120 and the remaining traders are informed traders (80). Consider a base price of \$10, as shown in the fourth row of Table 3. First, the fraction of discretionary trading (q^*) is equal to 0.7169 (fourth column), which implies that 72% of the 120 liquidity traders act as discretionary traders (=86.03, as shown in the fifth column) while the remaining 28% act as nondiscretionary traders.[17] Therefore, in each one of the m (=10) periods, 2.8% (=3.40, as shown in the last column) act as nondiscretionary liquidity traders. The (per dollar) adverse selection commission in the period of concentrated trading is 0.0051 (third column) whereas the (per dollar) adverse selection commission in the other nine periods is 0.0247 (second column).

In contrast, if the base price is equal to \$100, as shown in the last row of the table, a smaller fraction of the liquidity traders act as discretionary traders ($q^* = 35\%$). The (per dollar) adverse selection commission is 0.0077 in the period of concentrated trading and 0.0189 in the remaining regular periods.

Besides adverse selection commissions, liquidity traders also incur DRC of \$0.0625, on average, in a regular period and much lower DRC (as shown in the last row of Table 2) in the period of concentrated trading. Finally, discretionary liquidity traders also incur monitoring costs. The total per dollar expected transaction costs (incurred by all liquidity traders) can be computed for a given base price (P_0), as shown in Eq. (12). Table 4 shows the total equilibrium expected transaction costs (last column) at various base prices and Fig. 3 graphs the transaction costs as they vary with the base price.

It can be seen both from Table 4 and Fig. 3 that the transaction cost function [$TC(P_0)$] can be minimized by choosing an appropriate base price (P_0). In this case, the optimal base price is \$53 and the (per-dollar) transaction costs incurred at this base price are 3.1373. In contrast, had the base price been \$100 (last row), the (per dollar) transaction cost would have been 3.1923. This translates into a saving of 1.75%.[18]

We are also interested in finding out whether the optimal price is a global minimum or not. As the base price increases above \$53, the transaction cost function increases monotonically. No feasible solution exists beyond a base price of \$100. Therefore, the minimum at \$53 is a global minimum. In general, one cannot be sure whether the optimal price is a global minimum or not because we are employing numerical

[17] For convenience, we allow for fractional number of liquidity traders.

[18] It might seem as if there is not much difference in transaction costs at the optimal price level of \$53, where $TC(P_0) = 3.1373$, and a high price level of \$100, where $TC(P_0) = 3.1923$. Note that the expected transaction cost is a per dollar measure. This implies that an investment of \$100 when the base price is \$53 results in an absolute cost of $3.1373 \times 100 = \$313.73$. In contrast, had the base price been \$100, the absolute transaction costs would have been $3.1923 \times 100 = \$319.23$. Thus, holding the base price at \$53 results in a saving of \$5.50 (=1.75% of \$313.73) for 120 liquidity traders.

Table 4
Optimal price tradeoffs

This table shows total (expected) transaction costs incurred by all liquidity traders as a function of the base price. $P_0 \equiv$ base price, $q^* \equiv$ fraction of liquidity traders who choose to act as discretionary traders, $a_h \equiv$ adverse selection related commissions in a regular period, $a_l \equiv$ adverse selection related commissions in the period of concentrated trading, $T_D \equiv$ number of discretionary traders, and $T_{ND} \equiv$ number of nondiscretionary traders in each period. We assume that, in a continuum of increasing costs, the qth percentile liquidity traders faces a monitoring cost, $C(q) = f/[-\ln(q)]^{1/w}$, where $f > 0$ and $w > 1$. The parameters defining the numerical solution are as follows: (i) λ: the fraction of liquidity traders in the trading population (T), (ii) k: the volatility parameter, which specifies the standard deviation of the private information (δ) in $\sigma(P_0) = kP_0$, where P_0 is the base price, (iii) m: the number of periods, (iv) d: the tick size, and (v) (f, w): the monitoring cost parameter pair that defines the monitoring cost schedule. The parameters chosen for the simulation are (i) $\lambda = 0.6$, $T = 200$, (ii) $k = 0.02$, (iii) $m = 10$, (iv) $d = \$0.125$, and (v) $f = 0.0109$, $w = 1.6$.

	(Per dollar) expected transaction cost components			
Base price (P_0)	Adverse selection	Discreteness related	Monitoring cost	Total (per dollar) transaction costs
0.5	0.6109	6.3778	2.3976	9.3864
1	0.6844	2.4413	2.2569	5.3826
2	0.8163	1.2813	2.0725	4.1701
10	1.2093	0.3463	1.7195	3.2752
20	1.3597	0.1954	1.6207	3.1757
30	1.4357	0.1386	1.5751	3.1494
40	1.4850	0.1083	1.5469	3.1402
50	1.5215	0.0893	1.5267	3.1375
52	1.5278	0.0863	1.5232	3.1374
53	1.5309	0.0849	1.5215	3.1373
54	1.5339	0.0836	1.5199	3.1374
60	1.5509	0.0763	1.5107	3.1379
70	1.5762	0.0668	1.4971	3.1402
80	1.5996	0.0596	1.4848	3.1441
90	1.6233	0.0541	1.4725	3.1499
100	1.7123	0.0528	1.4272	3.1923

techniques to solve the model. However, given an upper bound on the feasible set of prices that a firm can consider, even a local minimum over a reasonable range of prices would suffice.[19]

[19] Our only concern is that the global minimum could be a corner solution because $TC(P_0)$ may go to 0 when P_0 approaches infinity. We have two comments to make. First the stock price level is bounded by the economic value of the firm (there must be at least one share), which rules out infinite values for P_0. Second, note that if market makers are risk averse or face wealth constraints, the breakeven commission charged by the market maker would increase at a faster rate than predicted by our model, which has risk neutral market makers. In such a setting, $TC(P_0)$ would not go to zero as P_0 approaches infinity, and an interior global optimum would be realized. We can, therefore, focus on the local minimum and interpret our model under the restriction that the base price has to be less than some upper bound.

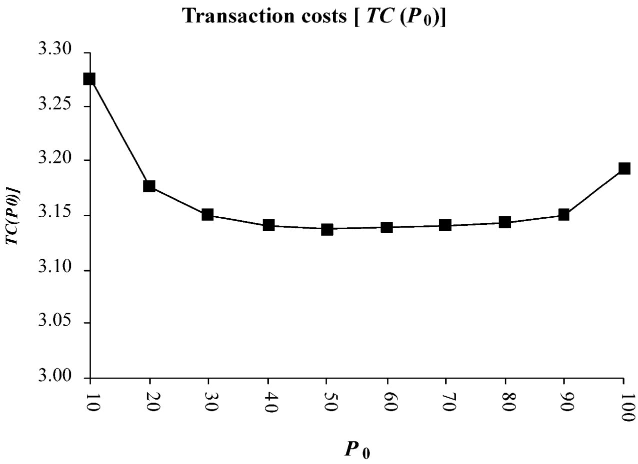

Fig. 3. The transaction costs function $[TC(P_0)]$ is shown as a function of the base price P_0. The transaction costs are made up of the sum of adverse selection related commissions and discreteness related commissions incurred by all liquidity traders in all periods as well as the (cumulative) monitoring costs incurred by all discretionary liquidity traders. At a base price of approximately \$53, the transaction costs are minimized.

4.3. Tradeoffs in the optimal price

The key feature of the model is that the base price (P_0) affects the economic significance of savings in execution costs accruing to potential discretionary traders. While the base price does not affect the (per dollar) adverse selection commissions, it reduces the economic significance of the fixed cost component – discreteness related commissions. Thus, (per dollar) execution costs depend on the base price (P_0). And, therefore, the amount of discretionary trading (q^*) depends on the base price (P_0). At lower base prices, there is greater discretionary trading because of the economic significance of *DRC*. Conversely, there is lesser discretionary trading at higher base prices. This dependence of the distribution of trades (across time) on the base price, in turn, affects total transaction costs incurred by traders across all the periods. In other words, total transaction costs depend on the base price (P_0).

To get a better insight of the tradeoffs in the optimal price, we rearrange the first two terms of the transaction cost function described in Eq. (12) as follows:

$$\begin{aligned} TC(P_0) &= [(m-1)a_{\mathrm{h}}T_{\mathrm{ND}} + a_{\mathrm{l}}(T_{\mathrm{D}} + T_{\mathrm{ND}})]/P_0 \\ &+ [(m-1)(d/2)T_{\mathrm{ND}} + E(DRC)_D(T_{\mathrm{D}} + T_{\mathrm{ND}})]/P_0 \\ &+ f(\lambda T)\Gamma((w-1)/w)\{1 - GAMMADIST[-\mathrm{Ln}(q^*)]\} \end{aligned} \tag{13}$$

$=$ sum of adverse selection commissions in m periods

$+$ sum of DRC in m periods

$+$ (cummulative) monitoring costs.

$TC(P_0)$, as described in Eq. (13) consists of three terms expressed in per dollar amounts. The first term is the sum total of (per dollar) adverse selection commissions paid by all liquidity traders in all m periods (for clarity, we refer to the sum total of adverse selection commissions as the adverse selection *component*). The second term is the sum total of (per dollar) DRC incurred by all liquidity traders in all m periods (again, for clarity, we refer to this as the *DRC component*). Finally, the third term indicates the (cumulative) monitoring costs incurred by discretionary liquidity traders (see Appendix C for a derivation of the monitoring cost *component*).

Table 4 shows the three components at different base prices. It can be seen (in the second column of Table 4) that the adverse selection component increases as the base price increases, whereas the DRC component (third column) and the monitoring cost component (fourth column) decrease with the base price. The optimal base price of \$53 strikes the right balance between these components.

The result can be explained with the help of Table 3. As the base price increases, fewer liquidity traders act as discretionary traders (q^* is lower) because the economic significance of savings in DRC is lower. The reduction in discretionary trading has the following effects. First, adverse selection commissions in the period of concentrated trading increase (third column in Table 3). Second, fewer traders benefit from trading in the period of concentrated trading (fifth column in Table 3). Third, more liquidity traders trade in regular periods (last column in Table 3). Fourth, the adverse selection commissions in the regular period decrease (second column in Table 3). However, they are still higher than in the period of concentrated trading. The net effect is that the adverse selection component of $TC(P_0)$ increases with the base price (P_0).

Now consider the DRC component of $TC(P_0)$. Less concentrated trading at higher base prices implies that fewer liquidity traders incur the low DRC in the period of concentrated trading. Also, more liquidity traders trade in the other periods where DRC is, on average, higher. Therefore, both effects work toward increasing (dollar) DRC. However, (dollar) DRC does not increase as fast as the base price and the (per dollar) DRC component decreases in the base price, unlike the adverse selection component.

Finally, the monitoring cost component decreases with the price level because there is less concentrated trading at higher price levels and fewer liquidity traders incur monitoring costs. There exists a tradeoff between an increasing adverse selection cost component and decreasing DRC and monitoring cost components of the transaction cost function. This tradeoff results in an interior optimum.

The intuition of the model can also be explained as follows. A lower base price induces more liquidity traders to act as discretionary traders. This is beneficial because greater discretionary trading results in a lower adverse selection component. However, a lower base price also has adverse cost implications. First, the *DRC* component increases and higher (cumulative) monitoring costs are incurred because more liquidity traders act as discretionary traders. The optimal price, which results in an optimal amount of discretionary trading, is the one equating the marginal adverse selection component on the one hand to the sum of the marginal *DRC* and the marginal monitoring components on the other hand.[20]

4.4. Robustness checks

To check the robustness of our results, we numerically solved the model for various parameter values.[21] The basic parameters are the liquidity parameter (λ), the volatility parameter (k), the trading horizon (m), and the tick size (d). Given specific values for λ, k, m, and d, we find that the existence of an interior local optimum of the transaction cost function depends on monitoring cost parameter pair (f, w). We show that by appropriately choosing values for (f, w), one can demonstrate the existence of an interior (local) optimal price for a wide range of reasonable parameter values for $\{\lambda, k, m, d\}$.[22]

The results are presented in Table 5. In Panel A of Table 5, we consider two polar cases of the liquidity parameter (λ). Consider the case where $\lambda = 0.1$. By appropriately choosing $(f, w) \equiv (0.0119, 2.1)$, we show the existence of an optimal price (\$32) when λ is as low as 0.1. On the other hand, by choosing $(f, w) \equiv (0.0065, 2.2)$, an optimal price exists (\$71) when λ is as high as 0.9.

[20] A final point is in order. At first glance, it might seem that the results of our model could be obtained by explicitly introducing the asset price level and endogenizing discretionary trading in the Admati and Pfleiderer (1989) model. However, our model has an additional feature – price discreteness. The exchange-mandated discrete pricing restriction plays an important role in our model. To see this, observe Eq. (10) that describes the savings in execution costs. Note that in the absence of discrete prices, savings in executions costs arise purely because of savings in adverse selection commissions, $[(a_{\mathrm{h}} - a_{\mathrm{l}})]/P_0$, and not because of savings in *DRC*, $[d/2 - \mathrm{E}(DRC)_{\mathrm{D}}]/P_0$. As shown in Appendix A, the saving, $[(a_{\mathrm{h}} - a_{\mathrm{l}})]/P_0$, is independent of the price level. Therefore, q^*, which is a function of the savings, is independent of the base price in this set up [see Eq. (11)]. As a consequence, in a model without discrete prices, the total transaction cost function would be independent of the price level. Unlike our model, there would be no optimal price.

[21] We find that, for some parameter sets, there exist multiple solutions for the endogenous variables (a_{h}, a_{l}, and q^*), depending on the initial seed values. To obtain the results in the paper, use initial seed values close to the given solution of a_{l} and a_{h}.

[22] For some parameter sets, the transaction cost function, $TC(P_0)$, is monotonically decreasing in P_0, and no interior optimal price exists.

Table 5
Robustness checks

This table presents the optimal base price for different parameter values. We assume that, in a continuum of increasing costs, the qth percentile liquidity traders faces a monitoring cost, $C(q) = f/[-\ln(q)]^{1/w}$, where $f>0$ and $w>1$. The parameters defining the numerical solution are as follows: (i) λ: the fraction of liquidity traders in the trading population (T), (ii) k: the volatility parameter, which specifies the standard deviation of the private information (δ) in $\sigma(P_0)=kP_0$, where P_0 is the base price, (iii) m: the number of periods, (iv) d: the tick size, and (v) (f, w): the monitoring cost parameter pair that defines the monitoring cost schedule. T is assumed to be equal to 200 in all cases.

Panel A. Liquidity parameter (λ)			
$\lambda=0.1$	$(f,w)\equiv(0.0119, 2.1)$	$(m,k,d)\equiv(10, 0.02, 0.125)$	$P_0^*=32$
$\lambda=0.9$	$(f,w)\equiv(0.0065, 2.2)$	$(m,k,d)\equiv(10, 0.02, 0.125)$	$P_0^*=71$
Panel B. Volatility parameter (k)			
$k=0.002$	$(f,w)\equiv(0.0015, 1.5)$	$(m,\lambda,d)\equiv(10, 0.60, 0.125)$	$P_0^*=98$
$k=0.02$	$(f,w)\equiv(0.0105, 1.6)$	$(m,\lambda,d)\equiv(10, 0.60, 0.125)$	$P_0^*=120$
Panel C. Trading horizon (m)			
$m=10$	$(f,w)\equiv(0.0105, 1.6)$	$(\lambda,k,d)\equiv(0.70, 0.02, 0.125)$	$P_0^*=35$
$m=11$	$(f,w)\equiv(0.0105, 1.6)$	$(\lambda,k,d)\equiv(0.70, 0.02, 0.125)$	$P_0^*=72$
Panel D. Tick size (d)			
$d=\$0.0625$	$(f,w)\equiv(0.0105, 1.6)$	$(m,k,\lambda)\equiv(10, 0.02, 0.60)$	$P_0^*=60$
$d=\$0.125$	$(f,w)\equiv(0.0105, 1.6)$	$(m,k,\lambda)\equiv(10, 0.02, 0.60)$	$P_0^*=120$

Similarly, in Panel B of Table 5, we demonstrate the existence of an optimal price for two polar cases of the volatility parameter. For $k=0.2\%$, we let $(f,\ w)\equiv(0.0015,\ 1.5)$ and show that an optimal price exists (\$98). For $k=2\%$, we let $(f,\ w)\equiv(0.0105,\ 1.6)$ and show that the optimal price is \$120. Panel C of Table 5 shows how the optimal price changes as the trading horizon (m) increases from 10 to 11. We find that the optimal price increases with the trading horizon. Finally, we demonstrate the interior optimal price for two different tick size regimes in Panel D of Table 5.

4.5. Optimal price behavior

Fig. 4 illustrates the results of numerical solutions. Panel A shows that the optimal price is increasing in the volatility parameter (k). Panels B, C, and D show that the optimal price is decreasing in the liquidity parameter (λ), and the monitoring cost parameters, f and w, respectively. Finally, Panel E shows that the optimal price is increasing in the tick size (d).

The relationship between the optimal price and the tick size is linear. Table 6 shows the optimal price for different tick sizes and the consequent effective tick

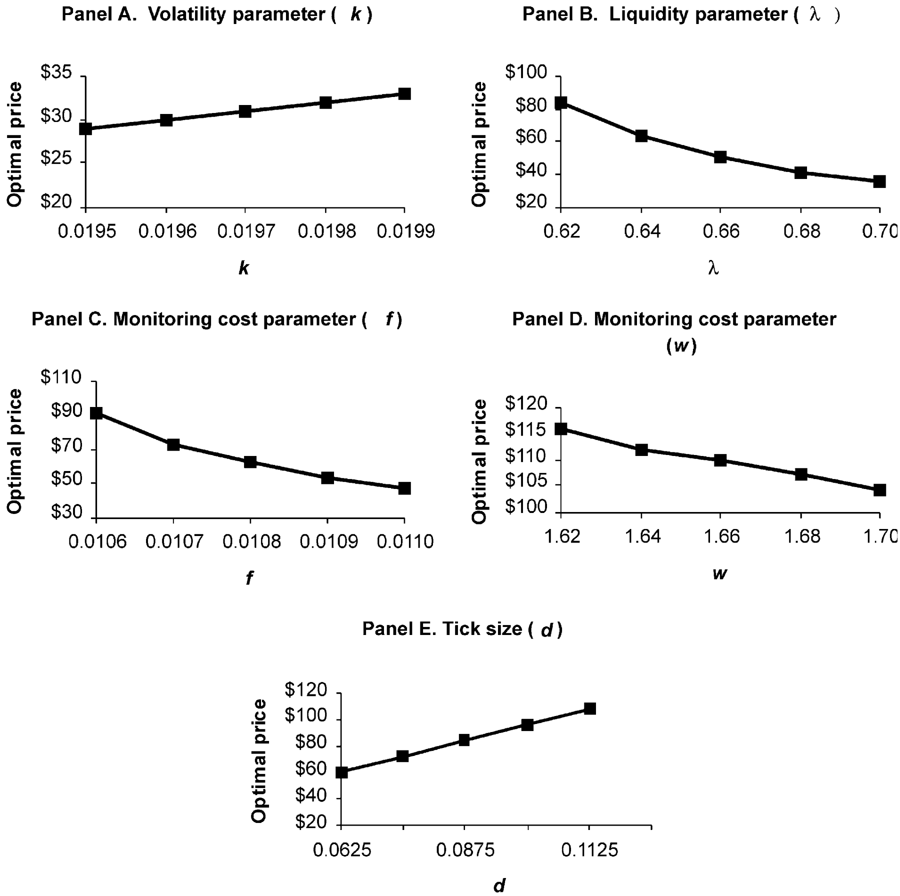

Fig. 4. The optimal price behavior. Panels A–E describe how the optimal price changes as one of the exogenous parameter changes in value. The optimal price is increasing in the volatility parameter (k), decreasing in the liquidity parameter (λ), monitoring cost parameters (f, w), and increasing in the tick size (d). Note that the relationship between the optimal price and the tick size is linear.

size (ratio of tick size to price). It can be seen that the effective tick size stays the same in all cases. In other words, given a set of parameters (other than the tick size), there exists an optimal effective tick size that minimizes transaction costs. As the tick size changes, the optimal price also changes proportionately, resulting in the same effective tick size.

What really matters is the effective tick size. In the U.S., because the tick size is fixed across all price levels (barring stock prices below $1), the optimal effective tick size can be transformed into an equivalent optimal price level. Furthermore, one can also interpret this model in terms of an optimal amount

Table 6
Tick size and optimal price

This table shows the optimal price behavior as the tick size changes. $P_0 \equiv$ base price, $q^* \equiv$ fraction of liquidity traders who choose to act as discretionary traders. We assume that, in a continuum of increasing costs, the qth percentile liquidity traders faces a monitoring cost, $C(q) = f/[-\ln(q)]^{1/w}$, where $f > 0$ and $w > 1$. The parameters defining the numerical solution are as follows: (i) λ: the fraction of liquidity traders in the trading population (T), (ii) k: the volatility parameter, which specifies the standard deviation of the private information (δ) in $\sigma(P_0) = kP_0$, where P_0 is the base price, (iii) m: the number of periods, (iv) d: the tick size, and (v) (f, w): the monitoring cost parameter pair that defines the monitoring cost schedule. The parameters chosen for the simulation are: (i) $\lambda = 0.6$, $T = 200$, (ii) $k = 0.02$, (iii) $m = 10$, and (iv) $f = 0.0105$, $w = 1.6$.

Tick size (d)	Optimal price (P_0^*)	Optimal effective tick (d/P_0^*)	Optimal amount of discretionary trading (q^*)	Total (per dollar) minimized transaction cost [$TC(P_0^*)$]
0.0625	60	0.0010	0.5183	3.0249
0.0750	72	0.0010	0.5183	3.0249
0.0875	84	0.0010	0.5183	3.0249
0.1000	96	0.0010	0.5183	3.0249
0.1125	108	0.0010	0.5183	3.0249

of discretionary trading (q^*).[23] The optimal effective tick size induces an optimal amount of discretionary trading. The last two columns show that the equilibrium amount of discretionary trading (q^*) and the minimized transaction cost (at the optimal price) are the same for all cases.

5. Empirical implications and evidence

Our theory is consistent with existing empirical evidence but it also leads to new empirical implications. We contrast the predictions of our theory with the predictions of the popular hypothesis, which states that splits allow a larger set of stockholders to buy round lots, thereby lowering the commissions paid and thus increasing the firms' stockholders base. We start by presenting empirical evidence consistent with both our theory and the popular hypothesis. We then present empirical evidence consistent with our model, but inconsistent with the popular hypothesis. Table 7 presents a list of empirical implications of our model and the popular hypothesis and the related evidence.

[23] Although q^* is endogenously determined, it is not a choice variable like the tick size or the base price.

Table 7
Empirical implications

This table presents a list of empirical implications and corresponding evidence of our theory and the popular hypothesis of stock splits.[a]

	Empirical implications		
	Our theory	Popular hypothesis	Evidence
Stock price before the split	Increase	Increase	Increase (Fama et al., 1969)
Announcement effects	Positive	Positive	Positive (Grinblatt et al., 1984)
Price level in U.S.	Constant	Constant	Constant (Angel, 1997)
Frequency of splits in Japan (relative to U.S.)	Lower	The same	Lower (Section 5.2)
Price level in Japan	Increase with time	Constant	Increase with time (Section 5.3)
Concentration of trades and price level	Negative relationship is possible	No relationship	Negative relationship (Section 5.4)
Concentration of trades after a split	Higher	The same	Higher (Anshuman and Goyal, 2001)
Stock return variability	Higher	The same	Higher (Ohlson and Penman, 1985)

[a] The popular hypothesis states that firms split their stocks to enable investors with binding wealth constraints to buy round lots; thereby reducing their transaction costs. By doing that the firm increases its stockholders' base.

5.1. Evidence consistent with our model and the popular hypothesis

Our model and the popular hypothesis predict that, other things constant, significant increases in stock price should precede stock splits. Both the theory and the hypothesis imply that announcements of splits would lead to positive stock market response.[24] Empirical evidence consistent with this implication has been documented in Fama et al. (1969). Positive announcement effects are documented in Grinblatt et al. (1984), and Eades et al. (1984). We turn now to present evidence consistent with our model and unrelated to the popular hypothesis.

5.2. Stock distributions and the minimum tick size

The pattern of stock distributions varies significantly between the U.S. and Japan. Relative to Japanese firms, U.S. corporate managers seem to split their

[24] Note, this conclusion depends on the extent to which the market is surprised by the firm's decision to split. This situation can arise if (i) the firm is more knowledgeable about the composition of liquidity traders, (ii) the firm has a better estimate of the costs of implementing a split, and (iii) the firm (as compared to the market) can better anticipate changes in the premium associated with its own stock's liquidity. Under such circumstances, market participants would not be able to predict with certainty the timing or the magnitude of the split.

stocks much more frequently. Our model can help explain this puzzling difference by examining the minimum price variation rules in Japan. The minimum tick size is constant only over a range of stock prices and increases as the level of the stock price increases. For instance, the tick size is 1 yen for stock prices up to 2000 yen, 5 yen for stock prices in the 2000–3000 yen range, 10 yen for stock prices in the 3000–30,000 yen range, 50 yen for stock prices in the 30,000–50,000 yen range, and so on. The effective tick size varies over the interval (0.0003, 0.0100) over all price levels. In contrast to the U.S., the effective tick size stays bounded within an interval and does not monotonically decrease with the stock price level. Our theory predicts that Japanese firms can allow the stock price level to drift without being too concerned about moving too far away from an optimal effective tick size. Unlike the U.S., Japanese firms need not maintain constant nominal prices. We, therefore, expect them to split their stocks less frequently than U.S. firms.

Fig. 5 (Panel A) displays the relative occurrence of stock dividends and splits in the U.S. during the period 1985–1999. The data used in this figure has been derived from the NYSE Fact Book (1994, 1999). There were a total of 2644 distributions during this period. Among these distributions, there were 362 (about 14%) distributions of size less than 25% whereas 2282 (about 86%) distributions qualified as stock splits. 1399 (about 53%) of the distributions were of size greater than 100%. During this period, U.S. firms have employed a greater proportion of stock splits as compared to stock dividends.

Fig. 5 (Panel B) presents similar information for the Japanese stock markets during the eight-year period 1983–1990. There were approximately 2982 stock distributions. Interestingly, the average split factor is only 17% (the minimum is 1%, and the maximum is 50%). There were 2799 distributions (about 94%) of size less than 25% and zero distributions of size greater than 100%. More importantly, only 183 distributions (about 6%) fell under the category of stock splits. In Japan, stock dividends seem to be much more prevalent than stock splits.[25]

5.3. Average stock price

Because the tick size is fixed in nominal terms at \$0.125, U.S. firms wishing to maintain a constant optimal (effective) tick size must keep their nominal stock price constant. Splits and reverse splits can be used to achieve this outcome. Indeed the empirical evidence indicates that, in spite of inflation, positive real interest rates, and significant risk premiums, the average nominal stock price in the U.S. during the last 50 years has been almost constant. Angel (1997) presents empirical evidence, which shows that the average share price on

[25] One must also note that institutional differences between the U.S. and Japan regarding stock distributions might provide an alternative explanation of the results reported here.

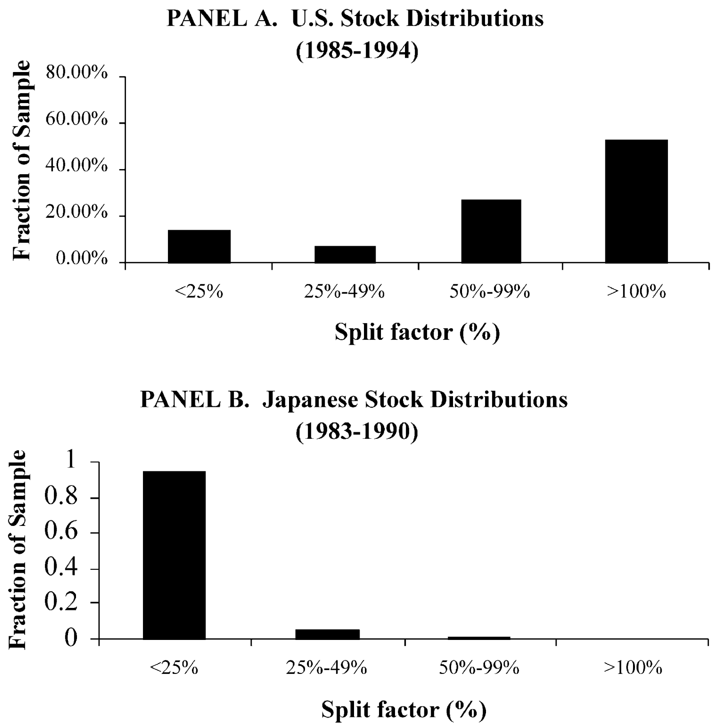

Fig. 5. Stock distributions in the U.S. and Japan. Panels A and B show the histogram of stock distributions in the U.S. (1985–1999) and Japan (1983–1990), respectively. The proportion of stock splits to stock dividends in Japan is much lower than in the U.S. [Source: Panel A has been constructed from data appearing in NYSE Fact Book (1994, 1999) and Panel B has been constructed from data appearing in Ide (1994).]

the NYSE has changed from \$39.86 in 1951 to \$37.27 in 1991. The average share price on the NYSE has mostly hovered very close to the \$40 mark during the period 1951–1991.

As mentioned above, Japanese firms, in contrast, can maintain a long-run stable effective tick size even in the presence of rapid stock price increases. Thus, Japanese firms need not maintain constant nominal stock prices to attain their optimal effective tick size. In the presence of positive inflation, real interest rates, and positive risk premiums, one should expect to see a rising trend in the average stock price over an extended period of time. The evidence on the distribution of stock prices on the first section of the Tokyo Stock Exchange over the period 1950–1990 is consistent with this prediction. Ide (1994) reports that the percentage of stocks with market value less than 500 yen has decreased from 99% in 1950 to about 8% in 1990. During the same period the percentage of stocks between 1000 and 2000 yen has risen from zero to 31% and the percentage of stocks above 2000 yen has increased from zero to 17%.

This rising trend is also apparent from the increase in the average share price from 74 yen in 1950 to 1577.50 yen in 1990.

5.4. Coefficient of variation of per period trading volume and price levels

Our model implies that different firms would maintain their stock at different optimal levels depending on the relevant values of the parameter set $\{\lambda, k, m, d, f, w\}$. Interestingly, an optimal price is also associated with an equilibrium amount of discretionary trading (q^*). Therefore, different firms (preferring different optimal prices) would experience differing degrees of discretionary trading (or concentrated trading), depending on their parameter values.

In this section we empirically examine the cross-sectional relationship between price levels of firms and the degree of concentrated trading in their stock. For this purpose, we study a sample of intraday transaction data for 2173 stocks trading on the NYSE during August 1993. The raw data consist of time-stamped transactions as well as quote data obtained from the TAQ data set. For transactions, price and volume data are available, while for quotes, bid and ask quotes are available. To allow for a homogenous market regime, the opening and closing sessions of trading have not been used in our empirical tests. Thus, the sample consists of data from 22 trading days between 10:00 a.m. and 4:00 p.m.

For each stock, we computed the average transaction price (*PRICE*) based on all transactions for that stock in the 22-day trading period. *PRICE* represents the stock price level. For the entire sample of stocks, *PRICE* had a mean of \$27.90, a minimum value of \$0.091825, and a maximum value of \$6830.49.

For our sample data on the NYSE, the tick size is equal to \$1/16 for stocks trading between \$0.25 and \$1.00, and further reduces to \$1/32 for stocks priced below \$0.25. We excluded transaction data for all stocks that had *PRICE* less than or equal to one dollar in order to restrict our test to a single tick size. Furthermore, we also excluded all stocks that had less than (or equal to) 100 transactions over the entire sample period. After these exclusions, our final sample had 1796 stocks.

For each stock in the sample, we aggregated trading volume over 5-, 15-, 30-, and 60-minute intervals. Over 22 trading days, aggregation of 5-minute trading volume results in 1584 observations (22 days $\times$ 6 hours $\times$ 12 5-minute intervals) for each stock. Similarly, each stock has 528 15-minute trading volume observations, 264 30-minute observations, and 132 60-minute observations.

VSTD denotes the standard deviation of the volume series of a stock. We ran a regression of *VSTD* on *PRICE*. Because *VSTD* may be higher for stocks with higher average trading volume, we added *VMEAN* as an independent variable in the regression, where *VMEAN* is the mean of the aggregated volume of the stock in question. The results of this regression are reported in Table 8. The first four columns show the results of the regression where volume is aggregated every 5, 15, 30, and 60 minutes, respectively. The evidence presented here indicates that volatility of aggregated volume is negatively related to the price level. Higher priced stocks have a smoother volume of trade.[26] To check for multicollinearity, we computed the correlation between *PRICE* and *VMEAN* and found it to be statistically insignificant. The Pearson correlation is equal to -0.0439 (p-value $= 0.0406$), which indicates that the regression specification does not suffer from multicollinearity.

Table 8 also presents results associated with an alternative proxy of the degree of concentrated trading, the Herfindahl Index. It is computed as follows. If $V05_i$ denotes the trading volume for a stock in a particular 5-minute trading period i, and *V05SUM* denotes the total trading volume for that stock over the entire sample period, then the Herfindahl Index (*HERF*) is given by $\Sigma_i(V05_i/V05SUM)^2$. The Herfindahl indices for 15-, 30-, and 60-minute aggregated volume data are constructed in a similar manner. Table 8 shows that the Herfindahl Index is (significantly) negatively related to the stock price level. These results imply that the volume series of higher priced stocks is smoother than that of lower priced stocks.

Next we investigate whether an inverse relationship between price level and the degree of concentrated trading is possible in the context of our model. To measure the degree of concentrated trading, we note that the per period trading volume in the m periods is not constant because the trading volume increases in the period of concentrated trading. We can show that the mean and standard deviation of per period (expected) trading volume is $[\lambda T/m + (1-\lambda)T]$ and $(1/m)^{0.5}[q^*\lambda T]$, respectively.[27] It follows that the coefficient of variation, which is defined as the ratio of the standard deviation to the mean, is given by

[26] In an earlier version of the paper we used intraday transaction data from 42 trading days during the months of March and April, 1985. We found very similar results to that reported in Table 8.

[27] The m trading periods can be classified into $(m-1)$ regular periods and one period of concentrated trading. In a regular period only nondiscretionary liquidity trading occurs along with informed trading. In contrast, in the period of concentrated trading, both nondiscretionary and discretionary liquidity trading occurs along with informed trading. The per period trading volume is therefore not constant across all periods, as shown in Fig. 2. Trading volume in a regular period is equal to $(1-q^*)\lambda T/m$ of nondiscretionary trades and $(1-\lambda)T$ of informed trades. In contrast, trading volume in the period of concentrated trading is equal to $q^*\lambda T$ of discretionary trades, $(1-q^*)\lambda T/m$ of nondiscretionary trades, and $(1-\lambda)T$ of informed trades. The mean of per period trading volume in $(m-1)$ regular periods and one period of concentrated trading is given by $[(m-1)/m][(1-q^*)\lambda T/m + (1-\lambda)T] + (1/m)\,[q^*\lambda T + (1-q^*)\lambda T/m + (1-\lambda)T]$, which simplifies to $[\lambda T/m + (1-\lambda)T]$. The variance of per period trading volume is given by $(1/(m-1))[(m-1)\{[(1-q^*)\lambda T/m + (1-\lambda)T] - [\lambda T/m + (1-\lambda)T]\}^2 + \{[q^*\lambda T + (1-q^*)\lambda T/m + (1-\lambda)T] - [\lambda T/m + (1-\lambda)T]\}^2]$, which simplifies to $(1/m)\,[q^*\lambda T]^2$. It follows that the standard deviation is $(1/m)^{0.5}[q^*\lambda T]$ and the coefficient of per period trading volume is given by $m^{0.5}[q^*\lambda]/[\lambda + (1-\lambda)m]$.

Table 8
Concentration of trading volume

This table shows the results of the following regressions:

$$VSTD = a + b\,PRICE + c\,VMEAN,$$

$$HERF = a + b\,PRICE + c\,VMEAN,$$

where $VSTD \equiv$ (timeseries) standard deviation of each stock's volume aggregated over a 5 (15, 30, and 60) minute interval, $PRICE \equiv$ average transaction price of each stock over the entire sample period, $VMEAN \equiv$ (timeseries) mean of each stock's volume aggregated over a 5 (15, 30, and 60) minute interval, and $HERF \equiv$ Herfindahl Index of the 5 (15, 30, and 60) minute volume series. The number of observations in all regressions is equal to 1796. Each cell contains the regression coefficient, the t-statistic, and the p-value, respectively.

	$VSTD = a + bPRICE + cVMEAN$				$HERF = a + bPRICE + cVMEAN$			
	(1) 5 min	(2) 15 min	(3) 30 min	(4) 60 min	(1) 5 min	(2) 15 min	(3) 30 min	(4) 60 min
R^2	0.6474	0.7255	0.7263	0.7172	0.0323	0.0404	0.0391	0.0361
INTERCEPT	4556.44	7602.27	10,076	5751.26	0.0306	0.0418	0.0577	0.0398
	14.292	13.177	10.025	11.068	33.665	45.902	58.545	40.924
	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
PRICE	−6.724	−6.906	−5.959	−7.465	−7.04E−06	−4.10E−06	−2.75E−06	−6.02E−06
	−8.465	−10.459	−9.821	−9.414	−3.103	−3.937	−4.616	−4.036
	0.0001	0.0001	0.0001	0.0001	0.0019	0.0001	0.0001	0.0001
VMEAN	1.122	0.8554	0.7480	1.101	−2.71E−07	−7.84E−08	−2.96E−08	1.17E−07
	53.069	58.936	57.688	59.444	−4.489	−3.422	−2.331	−3.367
	0.0001	0.0001	0.0001	0.0001	0.0001	0.0006	0.0199	0.0008

Table 9
Concentration of trading volume and price level

This table shows that a negative relationship between the degree of concentrated trading and the price level is possible in the context of the model. As λ increases, the optimal price increases. In contrast, the sample standard deviation and coefficient of variation of per period trading volume decrease with λ. $P_0 \equiv$ base price, $q^* \equiv$ fraction of liquidity traders who choose to act as discretionary traders. We assume that, in a continuum of increasing costs, the qth percentile liquidity traders faces a monitoring cost, $C(q) = f/[-\ln(q)]^{1/w}$, where $f > 0$ and $w > 1$. The parameters defining the numerical solution are as follows: (i) λ: the fraction of liquidity traders in the trading population (T), (ii) k: the volatility parameter, which specifies the standard deviation of the private information (δ) in $\sigma(P_0) = kP_0$, where P_0 is the base price, (iii) m: the number of periods, (iv) d: the tick size, and (v) (f, w): the monitoring cost parameter pair that defines the monitoring cost schedule. The parameters chosen for the simulation are: (i) $T = 200$, (ii) $k = 0.02$, (iii) $m = 10$, (iv) $d = \$0.125$, and (v) $f = 0.0105$, $w = 1.6$.

Liquidity parameter (λ)	Optimal price (P_0*)	Optimal effective tick size (d/P_0*)	Optimal amount of discretionary trading (q*)	Standard deviation of per period trading volume	Coefficient of variation of per period trading volume
0.70	35	0.0036	0.4543	404.44	0.2718
0.68	41	0.0030	0.4736	414.86	0.2625
0.64	63	0.0020	0.4985	407.12	0.2379
0.62	84	0.0015	0.5083	397.32	0.2255
0.60	120	0.0010	0.5183	386.79	0.2138

$m^{0.5}[q^*\lambda]/[\lambda + (1-\lambda)m]$. Note that both standard deviation and coefficient of variation are functions of q^*, which depends on the entire parameter set $\{\lambda, k, m, d, f, w\}$.

Table 9 shows a series of optimal prices and the associated coefficient of variation in per period trading volume. One can see that the optimal price changes as the liquidity parameter (λ) changes. Associated with each optimal price are the amount of discretionary trading (q^*), the standard deviation of per period trading volume and the coefficient of variation of per period trading volume. There is an inverse relationship between the price level and the standard deviation or the coefficient of variation in volume. This implies that our model is consistent with the empirically established negative relation between price and concentrated trading reported in this section.

5.5. Post-split coefficient of variation in trading volume

A stock split offers a natural experiment to empirically test our model. After a split, the only change that occurs is that of the price level. According to our model, the lower price level after a split should cause an increase in the amount of discretionary trading (q^*). From the discussion in the previous section, we can see that the standard deviation and coefficient of variation of per period trading volume are increasing in q^*. Our model, therefore, implies that a stock split should result in a higher standard deviation and coefficient of variation of per period trading volume. Anshuman and Goyal (2001) examine a sample of stock splits during 1993–1994 and find that, subsequent to stock splits, there is an overall increase in the coefficient of variation of trading volume aggregated over short intervals. This result is consistent with our model's prediction.

5.6. Post-split volatility of returns

Note that the periods of concentrated trading on the ask and bid sides of the market, in general, do not coincide because *DRC* on the ask side and bid side of the market are not identical.[28] Thus during periods of concentrated buying (selling), the average transaction price is biased toward the ask (bid) price. This bias causes significant price changes across the period of concentrated trading, thereby creating volatility. Given that there is greater amount of discretionary trading after a split, the above effect should be more pronounced after a split. Our model implies that the volatility of returns should increase (decrease) after a split (reverse split). Ohlson and Penman (1985), Lamoureux and Poon (1987), and Koski (1998) present evidence that is consistent with this implication of the model.

5.7. Decimalization of the minimum tick size

Our model suggests a linear relationship between the optimal price level and the tick size. This result has implications for the proposed decimalization of tick sizes on stock exchanges. If the tick size were to be reduced to 1 cent, the optimal price would drop by a factor of 12.5, i.e., a \$50 optimal price would drop down to \$4. This would mean that firms would undo the effect of the change in tick size by employing stock splits. According to our model, when firms believe that the new tick size is going to persist, they would employ a series of splits or a one-time large split to converge to lower trading range. This is an experiment that time will make possible. We believe that it is too early to interpret the current evidence (on stock split activity) either in favor of our model or against it.[29] More importantly, we note that exchanges and regulating bodies in the U.S. do not choose the effective tick size. They never did. Firms

[28] Asymmetric *DRC* on the ask and bid side arise because the true equilibrium price is, in general, asymmetrically located with respect to feasible discrete prices. See Anshuman and Kalay (1998) for a more detailed discussion of this issue.

[29] As pointed out by Angel (1997), there was a switch in price quoting rules from percentage of par to dollar pricing in 1915. With percentage of par pricing, the effective tick size is invariant to the price level and there were relatively few splits before 1915. After 1915, with dollar pricing, the effective tick size depended on the price level. However, it took about 14 years for the average stock price on the NYSE to drop from about \$90 to \$40, thereby doubling the effective tick size.

choose the effective tick size by splitting and reverse splitting the stocks. If we believe that insiders in firms know better than us as to what a stock's optimal effective tick size should be, any attempt to intervene via regulation is pointless.

6. Conclusion

This paper presents a theory that argues that splits improve liquidity. The driving force behind the model is the discrete pricing restriction on organized exchanges. Execution costs arising from discreteness vary over time. Liquidity traders have an incentive to time their trades to lower their execution costs. The asset price level affects their incentives because discreteness related costs are determined by a nominally fixed tick size. By splitting, a firm lowers its stock thereby increasing the incentives of liquidity traders to time their trades. The resulting concentration of trades reduces overall transaction costs incurred by liquidity traders. Our analysis also establishes the existence of an optimal price level that minimizes transaction costs. Stock split factors can be adjusted to reset the stock price level to the optimal level.

Appendix A. Derivation of Eq. (1)

The informed trader's profit conditional on observing δ is equal to $(\delta - a)$, where a is the ask commission charged by the market maker. The market maker infers that the informed trader would trade only when δ is greater than a. Therefore, he computes his expected losses to informed traders to be equal to $(1-\lambda)T\mathrm{E}[(\delta - a)|\delta > a]$. At the same time his expected profits from liquidity traders is equal to $(\lambda T/m)a$. Setting the market maker's expected profits to zero yields the equilibrium commission (a^*), i.e., a^* is given by the solution of the following equation:

$$-(1-\lambda)T\ \mathrm{Prob}\{\delta > a\}\mathrm{E}[(\delta - a)|\delta > a] + (\lambda T/m)a = 0. \tag{A.1}$$

The conditional expected value, $\mathrm{E}[(\delta) \mid \delta > a]$, given $\delta \equiv N(0, \sigma(P_0)^2)$, is equal to $\sigma(P_0)\ \{\phi(a/\sigma(P_0))/[1 - \Phi(a/\sigma(P_0)]\}$ (see Johnson and Kotz, p. 81), and noting that $\mathrm{Prob}\{\delta > a\} = [1 - \Phi(a/\sigma(P_0)]$, Eq. (A.1) may be simplified to: $-(1-\lambda)T\{\sigma(P_0)\phi(a/\sigma(P_0)) - [1 - \Phi(a/\sigma(P_0)]a\} + (\lambda T/m)a = 0$, as shown in Eq. (1), where $\phi(.)$ and $\Phi(.)$ represent the probability density function and the cumulative distribution function of the standard normal distribution, respectively.

Further, given $\sigma(P_0) = kP_0$, Eq. (1) reduces to: $-(1-\lambda)T\{kP_0\phi(a/kP_0) - [1 - \Phi(a/kP_0)]a\} + (\lambda T/m)a = 0$. Denoting (a/P_0) by x, this equation becomes $-(1-\lambda)T\{k\phi(x/k) - [1 - \Phi(x/k)]x\} + (\lambda T/m)x = 0$. Note that P_0 appears in this equation only through x. There is a unique solution to (x^*) to this equation, which implies that as P_0 changes, the variable a^* adjusts accordingly in such a way that (a^*/P_0) is always equal to x^*. In other words, the solution (a^*) is linear in P_0.

Appendix B

Lemma 1. The conditional distribution of DRC_{t+j} *given* DRC_t $\forall j > 1$, *is given by the wrapped normal distribution over the interval* (0, *d*).

For notational convenience, consider just the first two periods. Period 1 begins at time $t = 0$ and ends at time $t = 1$. Period 2 begins at time $t = 1$ and ends at time $t = 2$. As shown in Anshuman and Kalay (1998), the posted ask price is the nearest tick (d) greater than $P_0 + a^*$, where the base price (P_0) and the equilibrium ask commission (a^*) are common knowledge at time $t = 0$. It follows that the discreteness related commissions in Period 1 (DRC_1) are known to all traders at time $t = 0$, as shown below:

$$DRC_1 = \begin{cases} d - \mathrm{Mod}[P_0 + a^*, d] & \text{if } \mathrm{Mod}[P_0 + a^*, d] > 0, \\ 0 & \text{if } \mathrm{Mod}[P_0 + a^*, d] = 0. \end{cases}$$

It is more convenient to express the above formulation of DRC_1 as $\mathrm{Mod}[-(P_0 + a^*), d]$. Similarly, DRC_2 is given by $\mathrm{Mod}[-(P_0 + a^* + \delta), d]$, which is uncertain at time $t = 0$ because δ is going to be revealed at the end of Period 1. Hence, the distribution of DRC_2 is given by the distribution of the modulus of a nonzero mean normal random variable, $-(P_0 + a^* + \delta)$, which is $\mathrm{N}(\mu, \sigma^2)$, where $\mu = -(P_0 + a^*)$.

Mardia (1972) discusses the distribution of the modulus of a mean zero normal variable (referred to as the wrapped normal distribution). Extending his results, we get the probability density function of DRC_2 as

$$f(\theta; \mu) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{k=-\infty}^{k=+\infty} \exp\left(-\frac{1}{2} \frac{[\theta + dk) - \mu]^2}{\sigma^2} \right),$$
$$\text{where } \mu = -(P_0 + a^*) \quad \text{and } 0 \leqslant \theta < d.$$

Noting that knowledge of μ is equivalent to knowledge of $DRC_1 = \mathrm{Mod}(\mu, d)$, it follows that

$$f(\theta; DRC_1) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{k=-\infty}^{k=+\infty} \exp\left(-\frac{1}{2} \frac{[(\theta + dk) - DRC_1]^2}{\sigma^2} \right), \quad 0 \leqslant \theta < d. \tag{B.1}$$

The cumulative distribution function $F[\theta; DRC_1]$, can be computed by integrating each term of the above expression. This is the conditional distribution of DRC_2 given DRC_1. It follows that

$$F[\theta|DRC_1] = \sum_{k=-\infty}^{k=\infty} \left[\Phi\left(\frac{(\theta + dk) - DRC_1}{\sigma}\right) - \Phi\left(\frac{dk - DRC_1}{\sigma}\right)\right], \qquad \text{(B.2)}$$

where $\Phi(.)$ denotes the cumulative distribution function of the standard normal distribution. The conditional distribution of DRC_t given DRC_1 can be derived in a similar manner. The only difference is that the underlying normal random variable is $\sum_{1,\dots,t}(\delta)$, which is also normally distributed but has a much larger variance. The conditional distribution of DRC_t given DRC_1 would therefore be wrapped normal. In general, the above result applies for the conditional distribution of DRC_{t+j} given information at time t for all $j > t$. We refer to the conditional distribution as $F_{t+j}(.|DRC_t)$.

Proof of Proposition 1. We develop the optimal timing strategy of discretionary traders. The total execution costs depend on adverse selection commissions (a_l) and DRC. Discretionary traders can compare the execution costs in the current period ($a_l + DRC$) with the $\mathrm{E}\{a_l + DRC\}$ upon deferring trades. Note that a_l is minimized when ex-ante expected DRC are minimized. Hence, minimizing ($a_l + DRC$) is equivalent to minimizing DRC.[30]

Let the realized DRC in any period (s) be denoted by $DRC_s = z_s d$, $0 < z_s \leqslant 1$. Let the discretionary traders' trading (deferring) strategy be described by a set of cutoffs (α_s, $0 < \alpha_s \leqslant 1$, $s = 1, \dots, m$). Discretionary traders defer their trades whenever the realized DRC in any period (s) is such that $z_s > \alpha_s$.

Let $[DRC_{t,\dots,m}|DRC_{t-1} = z_{t-1}d]$ denote the (conditional) expected DRC when $(t-1)$ periods have elapsed, but period t has not yet begun. The discretionary trader has to trade in any one period between t and m. Note that discretionary traders would reach period t only if the realized DRC in all previous periods was greater than the corresponding cutoffs. More importantly, the expected DRC would now depend on the cutoffs $\alpha_{t,\dots,}\alpha_m$ and not on any of the previous cutoffs. It follows that

$$\begin{aligned} &E\,[DRC_{t,\dots,m}\,|\,DRC_{t-1} = z_{t-1}d] \\ &\quad = F_t(\alpha_t\,|\,DRC_{t-1} = z_{t-1}d)\mathrm{E}\{z_t d|0 \leqslant z_t \leqslant \alpha_t, DRC_{t-1} = z_{t-1}d\} \\ &\qquad + (1 - F_t(\alpha_t|DRC_{t-1} = z_{t-1}d))\mathrm{E}\{DRC_{t+1,\dots,m}|DRC_{t-1} = z_{t-1}d\}. \end{aligned} \qquad \text{(B.3)}$$

Note that α_t appears in $F_t(.)$ and $\mathrm{E}\{z_t d\}$, but does not appear in the term $\mathrm{E}\{DRC_{t+1,\dots,m}]$. The latter is a function of the cutoffs, $\alpha_{t+1,\dots,}\alpha_m$. Using the results for $F_t(.)$ in Eq. (B.2) and the result in Lemma 2 (stated below) Eq. (B.3)

[30] Since a_l is a function of q^*, which is a function of the savings form timing trades, a_l depends on the $\mathrm{E}(DRC)_\mathrm{D}$ from the overall dynamic strategy. Hence a_l is a function of all the optimal cutoffs from time $t = 1$ to m. As argued earlier (see footnote 13), a_l gets minimized when $\mathrm{E}(DRC)_\mathrm{D}$ gets minimized. Therefore, we can solve for the cutoffs by applying the dynamic optimization only with respect to DRC.

can be further simplified as

$$
\begin{aligned}
&\mathrm{E}[DRC_{t,\dots,m}|DRC_{t-1} = z_{t-1}d] \\
&= \sum_{k=-\infty}^{k=\infty} \sigma\left[\phi\left(\frac{dk - z_{t-1}d}{\sigma}\right) - \phi\left(\frac{\alpha_t d + dk - z_{t-1}d}{\sigma}\right)\right] \\
&\quad + (z_{t-1}d - dk)\left[\Phi\left(\frac{\alpha_t d + dk - z_{t-1}d}{\sigma}\right) - \Phi\left(\frac{dk - z_{t-1}d}{s}\right)\right] \\
&\quad + \left(1 - \sum_{k=-\infty}^{k=\infty}\left[\Phi\left(\frac{\alpha_t d + dk - z_{t-1}}{\sigma}\right) - \Phi\left(\frac{dk - z_{t-1}}{\sigma}\right)\right]\right) \\
&\qquad \mathrm{E}\{DRC_{t+1,\dots,m}|DRC_{t-1} = z_{t-1}d\}.
\end{aligned} \tag{B.4}
$$

Differentiating Eq. (B.4) with respect to α_t, and noting that the term $\mathrm{E}\{DRC_{t+1,\dots,m}\}$ does not contain α_t, we get upon simplification the following first order condition and second order condition (which ensures minimization):

$$\alpha_t^* : \alpha_t d - \mathrm{E}\{DRC_{t+1,\dots,m} \,|\, DRC_{t-1} = z_{t-1}d\} = 0, \tag{B.5}$$

$$d > 0. \tag{B.6}$$

Expansion of the term $\mathrm{E}\{DRC_{t+1,\dots,m} \,|\, DRC_{t-1} = z_{t-1}d\}$ in Eq. (B.5) results in the equation stated in Proposition 1 (after noting that the conditioning variable should be $DRC_t = z_t d$ at the beginning of period t).

This first order condition holds for all periods except the last period, where the constraint $\alpha_m^* = 1$ holds. Since the first order condition determining α_{m-1}^* depends on α_m^*, the problem can be solved using a dynamic programming procedure that solves the cutoffs in a sequential manner starting from the last period and going backward to the first period. In other words, Eq. (B.5) has to be solved recursively from $t = (m-1), \dots, 1$ to determine the cutoffs $\alpha_{m-1}^*, \dots, \alpha_1^*$, using the constraint $\alpha_m^* = 1$. □

Lemma 2

$$
\begin{aligned}
&\mathrm{E}\{z_t d \,|\, 0 \leqslant z_t \leqslant \alpha_t, DRC_{t-1} = z_{t-1}d\} \\
&= \frac{\sum_{k=-\infty}^{k=\infty} \sigma\left[\phi\left(\frac{dk - z_{t-1}d}{\sigma}\right) - \phi\left(\frac{\alpha_t d + dk - z_{t-1}d}{\sigma}\right)\right] + (z_{t-1}d - dk)\left[\Phi\left(\frac{\alpha_t d + dk - z_{t-1}d}{\sigma}\right) - \Phi\left(\frac{dk - z_{t-1}d}{\sigma}\right)\right]}{\sum_{k=-\infty}^{k=\infty}\left[\Phi\left(\frac{\alpha_t d + dk - z_{t-1}}{\sigma}\right) - \Phi\left(\frac{dk - z_{t-1}}{\sigma}\right)\right]},
\end{aligned}
$$

where $\phi(.)$ stands for the standard normal density function and $\Phi(.)$ stands for the cumulative standard normal.

The (conditional) probability density function of the $DRC_t = z_t d$ is given in Eq. (B.1) and the (conditional) cumulative distribution function is given in

Eq. (*B.2*). *Using these results, it follows that*

$$\mathrm{E}\{z_t d \mid 0 \leqslant z_t \leqslant \alpha_t,\ DRC_{t-1} = z_{t-1} d\}$$

$$= \frac{\int_0^{\alpha_t} \theta f(\theta)\,\mathrm{d}\theta}{\int_0^{\alpha_t} f(\theta)\,\mathrm{d}\theta} = \frac{\sum_{k=-\infty}^{k=+\infty} \int_0^{\alpha_t d} \frac{1}{\sigma\sqrt{2\pi}} \theta \mathrm{e}^{-(1/2)[(\theta+dk-z_{t-1}d)/\sigma]^2}}{\sum_{k=-\infty}^{k=+\infty} \left[\Phi\left(\frac{\alpha_t d + dk - z_{t-1}}{\sigma}\right) - \Phi\left(\frac{dk - z_{t-1}}{\sigma}\right)\right]}.$$

Using properties of the normal distribution, the result in Lemma 2 follows.

Appendix C. Derivation of cumulative monitoring costs

The cumulative monitoring cost component of the total transaction cost function is given by

$$\int_0^{q^*} C(q)\lambda T\,\mathrm{d}q = \int_0^{q^*} \frac{f\lambda T}{[-\mathrm{Ln}(q)]^{1/w}}\,\mathrm{d}q.$$

Let $s \equiv -\mathrm{Ln}(q)$. Then, the expression simplifies to $-(f\lambda T)\int_{\infty}^{Ln(q^*)} s^{-1/w}\mathrm{e}^{-s}\,\mathrm{d}s = (\lambda T) f \Gamma((w-1)/w)\{1 - GAMMADIST[-Ln(q^*))]\}$, where $\Gamma(.)$ denotes the gamma function and $GAMMADIST$ denotes the cumulative distribution function of the standard gamma distribution with parameter $[(w-1)/w]$. Note $w > 1$ is required for $\Gamma((w-1)/w)$ to be defined properly and the integration to hold.

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